Solutions to problems in Ch.10

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1 Problem 10.1

Claim 1 (Claim 10.1.1)**.** *Let $\pi : \bar{\mathcal{M}} \to \mathcal{M}$ be elementary and $\pi(\bar{U}) = U$. If a putative iteration of $\mathcal{M}$ of length $\alpha + 1$ is an iteration, then a putative iteration of $\bar{\mathcal{M}}$ of length $\alpha + 1$ is an iteration.*

Proof of claim: We recursively define, for $\xi \leq \alpha + 1$, $\pi_\xi : \bar{\mathcal{M}}_\xi \to \mathcal{M}_\xi$ by setting $\pi_{\xi+1}([f]_{\bar{U}_\xi}) = [\pi_\xi(f)]_{U_\xi}$, π_λ where λ is a limit to be the direct limit map given by $\pi_\lambda(\bar{i}_{\xi,\lambda}(f)) = i_{\xi,\lambda} \cdot \pi_\xi(f)$.

Then the final map $\pi_{\alpha+1} : \bar{\mathcal{M}}_{\alpha+1} \to \mathcal{M}_{\alpha+1}$ witness the wellfoundedness of $\bar{\mathcal{M}}_{\alpha+1}$. $\square$

Let $\mathcal{T}$ be a putative iteration of $\mathcal{M}$ of length $\alpha+1$. We pick θ large enough, X a ctm, and $\pi : X \to V_\theta$ that is elementary w.r.p the language with predicate $\in, \mathcal{T}, \mathcal{M}$. Let $\pi(\bar{\mathcal{T}}) = \mathcal{T}$, $\pi(\bar{\mathcal{M}}) = \mathcal{M}$. Then

$$X \models \bar{\mathcal{T}} \text{ is a putative iteration of } \bar{\mathcal{M}}$$

This is an absolute statement and hence $\bar{\mathcal{T}}$ is a putative iteration of $\bar{\mathcal{M}}$, as $\bar{\mathcal{T}} \in X$, its length is less than ω_1. By Claim 10.1.1 and the assumption in the problem we have $\bar{\mathcal{T}}$ is an iteration of $\bar{\mathcal{M}}$, i.e. $\bar{\mathcal{M}}_{\alpha+1}$ is well-founded. Hence $X \models \bar{\mathcal{T}}$ is an iteration of $\bar{\mathcal{M}}$. By elementariness and passing to V_θ, $\mathcal{T}$ is an iteration of $\mathcal{M}$. This concludes the proof that $\mathcal{M}$ is iterable.

cf. Lemma 2.4, Lemma 2.5 in John Steel's note on Itreated Ultrapowers.

2 Problem 10.2

(a) We show by an induction on $\alpha \in Ord$ that

$$M_\alpha = \{\pi_{0,\alpha}(f)(a) \mid a \in \{\kappa_\beta \mid \beta < \alpha\}^{<\omega}, f : [\kappa]^{|a|} \to M_0\}$$

The base case $\alpha = 0$ is trivial.

Induction step for successor $\alpha + 1$:

$$\begin{aligned} M_{\alpha+1} &= Ult(M_\alpha, U_\alpha) \\ &= \{\pi_{U_\alpha}^{M_\alpha}(g)(\kappa_\alpha) \mid g \in M_\alpha^{\kappa_\alpha}\} \\ &= \text{by IH}, \{\pi_{\alpha,\alpha+1}(\pi_{0,\alpha}(f)(a))(\kappa_\alpha) \mid a \in \{\kappa_\beta \mid \beta < \alpha\}^{<\omega}, f : [\kappa]^{|a|} \to M_0^\kappa\} \\ &= \{\pi_{0,\alpha+1}(f')(a \cup \{\kappa_\alpha\}) \mid a \in \{\kappa_\beta \mid \beta < \alpha\}^{<\omega}, f' : [\kappa]^{|a|+1} \to M_0\} \end{aligned}$$

The final equation holds as $\pi_{\alpha,\alpha+1}(a) = (a)$ for $a \subseteq \{\kappa_\beta \mid \beta < \alpha\}$. This equation shows the $\subseteq$ direction of the desired result. The other side is obvious.

Induction step for limit λ:

$$\begin{aligned} x \in M_\lambda &\iff \exists \alpha < \lambda, \exists y \in M_\alpha, x = \pi_{\alpha,\lambda}(y) \\ &\iff \exists \alpha < \lambda, \exists a \in \{\kappa_\beta \mid \beta < \alpha\}^{<\omega}, \exists f : [\kappa]^{|a|} \to M_0^\kappa, (x = \pi_{\alpha,\lambda}(\pi_{0,\alpha}(f)(a))) \\ &\iff \exists \alpha < \lambda, \exists a \in \{\kappa_\beta \mid \beta < \alpha\}^{<\omega}, \exists f : [\kappa]^{|a|} \to M_0^\kappa, (x = \pi_{0,\lambda}(f)(a)) \end{aligned}$$

This concludes the proof.

This exercise shows that $M_\alpha = h_M(ran(\pi_{0,\alpha}), \{\kappa_\beta \mid \beta < \alpha\})$.

(b) As $\kappa_\alpha, \alpha \in Ord$ satisfy $\kappa_\alpha < \kappa_\beta$ if $\alpha < \beta$, it is unbounded in Ord.

For arbitrary sequence $(\kappa_{\alpha_\mu}, \mu < \lambda)$ where λ is a limit, we show that $\bigcup_{\mu<\lambda} \kappa_{\alpha_\mu} = \kappa_{\bigcup_{\mu<\lambda} \alpha_\mu}$. Let $\theta = \bigcup_{\mu<\lambda} \alpha_\mu$.

$\leq$ is immediate. For the other side, if $\gamma < \kappa_\theta$, then $\gamma = \pi_{\alpha_\mu,\theta}(\gamma')$ for some $\mu < \lambda$ and by the construction of direct limit. $\gamma' < \kappa_{\alpha_\mu}$ by elementarily. It follows that $\gamma' = \gamma$ and hence $\gamma < \kappa_{\alpha_\mu}$.

(c) For limit ordinal λ, for arbitrary $X \in \kappa_\lambda \cap M_\lambda$, we have that there is some $\alpha < \lambda$ s.t. $\pi_{\alpha,\lambda}(Y) = X$ and $Y \in U_\alpha$. We show that for all β s.t. $\alpha \leq \beta < \lambda$, we have $\kappa_\beta \in X$. Then $Z := \pi_{\alpha,\beta}(Y) \in U_\beta$. Hence $\kappa_\beta \in \pi_{\beta,\beta+1}(Z) \subseteq \pi(\beta,\gamma)(Z) = X$ by normality of U_β. Hence $\kappa_\beta \in X$.

The statement is false for successor ordinals, as in this case the statement we are supposed to prove reduces to $\mathcal{M}_{\alpha+1} \models U_{\alpha+1}$ is principal.

(d) The statement contains an error, μ should be λ.

To see the statement makes sense, it follows from the proof of (b) that $\kappa_\lambda = \lim_{\beta<\lambda} \kappa_\beta$. We have $\kappa_\beta \leq |\beta| \cdot 2^\kappa < \lambda$ by a function counting argument, and hence $\kappa_\lambda = \lim_{\beta<\lambda} \kappa_\beta \leq \lambda$. This means $\kappa_\lambda = \lambda$.

The fact that $U_\lambda \subseteq F_\lambda \cap \mathcal{M}_\lambda$ follows from (c) and (b). The otherside follows as U_λ is an ultrafilter in $\mathcal{M}_\lambda$ $\Box$

3 Problem 10.3

(a) $L[U] = L[\bar{U}]$ is standard exercise. To verify $L[U] \models \bar{U}$ is a measure on κ, $\kappa \in \bar{U}$ as $\kappa \in L[U]$ and $\kappa \in U$.

If $X_1, X_2 \in \bar{U} = L[U] \cap U$, then $X_1 \cap X_2 \in L[U] \cap U$. Upward closure and the property for complement is similarly verified.

If $(X_\alpha, \alpha < \mu) \in L[U]$ for some $\mu < \kappa$ and $X_\alpha \in L[U] \cap U$, then $\bigcap_{\alpha<\mu} X_\alpha \in U$ and $\bigcap_{\alpha<\mu} X_\alpha \in L[U]$, hence $\bigcap_{\alpha<\mu} \in \bar{U}$.

(b) Same as 10.2 (d), μ should be λ. By elementarity $\mathcal{M}_\lambda \models V = L[U_\lambda]$. Hence $\mathcal{M}_\lambda = L[U_\lambda]$. As by 10.2 (d) $U_\lambda = F_\lambda^{L[U]} \cap \mathcal{M}_\lambda = F_\lambda \cap \mathcal{M}_\lambda$, $\mathcal{M}_\lambda = L[F_\lambda]$.

4 Problem 10.4

(a) Let $\lambda > 2^\kappa$ be regular, we show that there is δ s.t. $\mathcal{M}_\lambda = J_\beta[F_\lambda]$. First $\mathcal{M}_\lambda \models V = L[\pi_{0,\lambda}(U)]$, it must be of the form $J_\beta[\pi_{0,\lambda}(U)]$ for some limit β.

Next, notice that problem 10.2 also works for iterable set model $\mathcal{M}$. Hence we have $\pi_{0,\lambda}(U) = F_\lambda \cap \mathcal{M}_\lambda$ and hence $\mathcal{M}_\lambda = J_\beta[F_\lambda]$.

Hence two L^μ mouse can be coiterated as we can take $\lambda > max\{2^\kappa, 2^\lambda\}$ where κ, λ are the respective largest cardinal in $\mathcal{M}, \mathcal{N}$.

(b) Like in the proof of Claim 10.33, we construct the putative iteration of length $\gamma+1$, $(\mathcal{N}_\alpha, \theta_{\alpha,\beta} \mid \alpha \leq \beta \leq \gamma+1)$ of $\mathcal{N}_0 = J_{\bar{\alpha}}[\bar{U}]$ and construct the family of elementary embeddings $\sigma_\alpha, \alpha \leq \gamma+1$ into the iteration $(\mathcal{M}_\alpha, \pi_{\alpha,\beta} \mid \alpha \leq \beta \leq \gamma+1)$.

For successor step, we set $\sigma_{\alpha+1}(\theta_{\alpha,\alpha+1}(f)(\lambda_\alpha)) = \pi_{\alpha,\alpha+1}(\sigma_\alpha(f))(\kappa_\alpha)$. By the fact that $\mathcal{N}_0$ is a model of ZFC^-, this map can be elementary instead of just Σ_0 elementary. The limit case can be defined by commutativity of the diagram.

5 Problem 10.5

Let U be a normal measure on κ. We work in this inner model.

First we show the case for $\lambda < \kappa$: It suffice to show that for each λ and $X \in \mathcal{P}(\lambda)$,

$$|\{Y \in \mathcal{P}(\lambda) \mid Y \subseteq X \wedge Y <_{L[U]} X\}| \leq \lambda$$

Then we would have $o(<_{L[U]} |_{\mathcal{P}(\lambda)^2}) \leq \lambda^+$, this validates the conclusion.

Take $J_\alpha[U]$ large enough s.t. $\kappa, \lambda \in J_\alpha[U]$.

Consider the Skolem closure of $\lambda \cup \{\lambda\}$ w.r.t. the language with constant c_X for X, and collapse it to form $J_\beta[U']$ by condensation. Then $\pi : J_\beta[U'] \to J_\alpha[U]$ is elementary, $U' = \pi^{-1}[U]$, $|J_\beta[U']| = \lambda$, $\pi|_{\lambda+1} = id$ and $X = \pi(X')$ for some $X' \in J_\beta[U']$, we notice that actually $X = \pi(X)$ and thus $X \in J_\beta[U']$ as $\pi|_{\lambda+1} = id$.

Now we want to show that, assuming for $x, y \subseteq \lambda$, $x <_{L[U']} y$ iff $x <_{L[U]} y$, it holds that $\{Y \in \mathcal{P}(\lambda) \mid Y \subseteq X \wedge Y <_{L[U]} X\} \subseteq J_\beta[U']$, which finishes the proof. If $Y <_{L[U]} X$, there is γ s.t. $J_\alpha[U] \models rank_{<_{L[U]}}(Y) = \gamma$, then $J_\beta[U'] \models \exists Y' \subseteq X, rank_{<_{L[U']}}(Y') = \gamma$. By elementarity and the fact that $\pi|_{\lambda+1} = id$, such Y' is Y and hence $Y \in J_\beta[U']$.

Finally, we verify that for $x, y \subseteq \lambda$, $x <_{L[U']} y$ iff $x <_{L[U]} y$. By theorem 10.3 and Problem 10.4(b), $\mathcal{N}_0 = J_\beta[U']$ and $\mathcal{M}_0 = J_\alpha[U]$ are L^μ mice and hence by 10.4(a) they can be co-iterated, i.e. there is γ, γ' s.t. $\mathcal{M}_\lambda = J_\beta[F_\lambda], \mathcal{N}_\lambda = J_{\beta'}[F_\lambda]$. And hence $\mathcal{M}_0 \models x <_{L[U']} y$ iff $J_{\beta'}[F_\lambda], J_\beta[F_\lambda] \models x <_{L[F_\lambda]} y$ iff $\mathcal{N}_0 \models x <_{L[U]} y$ by elementariness and the fact that elements under λ are fixed under iteration.

Remark: In general, when we collapse the structure $(N, N \cap U)$, we can't make sure that the collapsed structure is $(N', N' \cap U)$, i.e. not necessarily elementary to $(L[U], U)$ for the predicate. Hence the condensation does not apply and though N' would be some $J_\beta[U']$, we have no idea about whether $U' = U \cap N'$. But in the case when U is a normal measure on κ, coiteration argument fits the gap.

The case when $\lambda \geq \kappa$ is similar to Godel's argument that $V = L$ implies GCH, for $x \subseteq \lambda$, the fact that $\lambda > \kappa$ means the collapsed structure of $(N =$

$h_{L[U]}(\lambda \cup \{x, U\}), U \cap N)$ to be elementary at the predicate as the collapsing map would be constant on $\kappa < \lambda$. Hence the condensation applies, this is not true for $\lambda < \kappa$.

6 Problem 10.6

The proof of Claim 10.22 is standard.

For 10.21 (d), we prove a more general theorem

Claim 2 (Claim 10.6.1)**.** *If $M_0, M_1 \models ZFC^-$ and $j : M_0 \to M_1$ is Σ_1 elementary and cofinal in M_1, then actually j is elementary.*

Proof of claim cf. Prop 5.1 in Kanamori: We show by an induction on the Levy hierarchy of formulas. Suppose $M_0 \prec_{\Sigma_n} M_1$, we show $M_0 \prec_{\Sigma_{n+1}} M_1$. It suffice to show that for Π_1 formula $\psi(x, \vec{y})$ and $\vec{a} \in M_0$, if $M_1 \models \exists x \psi(x, j(\vec{a}))$ then there is $b \in M_0$ s.t. $M_0 \models \psi(b, \vec{a})$.

By cofinalness, we find $c \in M_0$ s.t. $M_1 \models \exists x \in j(c) \psi(x, j(\vec{a}))$, by replacement (this is where $M_1 \models ZFC^-$ is used), $\exists x \in j(c)\psi(x, j(\vec{a}))$ is equivalent to a Π_n formula an hence $M_0 \models \exists x \in c\psi(x, \vec{a})$. We are done. □

I don't really understand why in the text book, the language in (c) and (d) is different.

7 Problem 10.7

We say a real x codes a structure $(M, \in, A)$ iff for $o(x) : n \mapsto 2n + 1$ and $e(x) : n \mapsto 2n$ $\pi(\omega, E_{o(x)}, A_{e(x)}) \cong (M, \in, A)$ where $E_{o(x)}$ denotes the standard coding of $o(x)$ as a well-founded relation and $n \in A_{e(x)} \iff e(x)(n) \neq 0$. π is the transitive collapse.

Claim 3 (Claim 10.7.1)**.** *The following relation is Δ^1_2:*

$$(x, y) \in A \iff x \text{ codes a premouse and its iteration up to } ||y||.$$

In the sense that for all n, $(x)_n : m \mapsto x(\Gamma(n, m))$ codes a premouse and

$$n_1 E_y n_2 \iff (x)_{n_2} \text{ codes an ultrapower of } (x)_{n_1}$$

Proof. First we show that x codes a premouse is Π^1_1. x codes a premouse iff $o(x) \in \mathsf{WF}$ and $(\omega, E_{o(x)}) \models ZFC^- + V = L +$there is a largest cardinal and say n_κ is the largest cardinal in $(\omega, E_{o(x)}$ $(\omega, E_{o(x)}, U_{e(x)}) \models U_{e(x)}$ is a non-trivial normal $<$ κ complete ultrafilter on κ.

$(\omega, E_{o(x)}) \models ZFC^- + V = L +$ there is a largest cardinal is arithmetical since the relation $(\omega, E_{o(x)}) \models \varphi(n_1 \dots n_m)$ is arithmetical. To analyze $(\omega, E_{o(x)}, U_{e(x)}) \models U_{e(x)}$ is a non-trivial normal $< \kappa$ complete ultrafilter on κ, for instance $(\omega, E_{o(x)}, U_{e(x)}) \models U_{e(x)}$ is $< \kappa$ complete. iff

$$\forall n(\pi_x(n) \in \pi_x(n_\kappa) \to \forall X \in [\omega]^\omega (\forall m \in X(m \in U_{e(x)} \wedge \pi_x(m) \in \pi_x(n)) \to \exists l \in U_{e(x)}(\bigcap \pi_x[X] = \pi_x(l)))$$

Where π_x is the Mostowski collapse. This is a Π^1_1 property, for the Mostowski collapse part see for instance Jech Proof of Lemma 25.25.

Next consider the following relation:

$$(x, y) \in Ult \iff y, x \text{ both code premice and } y \text{ codes the ultrapower of } x$$

First we notice the ultrapower equivalence relation for x is given by, for n_1, n_2 that are functions with domain ω in $(\omega, E_{o(x)})$ collapsed,

$$n_1 \equiv n_2 \iff \exists m(m \in U_{e(x)} \wedge \forall l(\pi(l) \in \pi(m) \to \pi(n_1)(\pi(l)) \in \pi(n_2)(\pi(l)))) \quad (*)$$

We have y codes the ultrapower of x iff there is $\equiv_x \subseteq \omega^2, f \in \omega^\omega$ s.t. $\equiv_x$ is an equivalent relation satisfying $*$, f respects $\equiv_x$ and is bijective $\omega \to \{n \in \omega \mid \pi_x(n) \text{ is a function with domain } \omega\}/\equiv_x$,

$$n_1 E_{e(y)} n_2 \iff \exists m(m \in U_{e(x)} \wedge \forall l(\pi(l) \in \pi(m) \to \pi(n_1)(\pi(l)) \in \pi(n_2)(\pi(l))))$$

and

$$n_1 \in U_{e(y)} \iff m(m \in U_{e(x)} \wedge \forall l(\pi(l) \in \pi(m) \to l \in U_{e(x)}))$$

And hence Ult is a Σ^1_1 relation.

Hence, let $(x)_n : m \mapsto x(\Gamma(n, m))$, $(x, y) \in A \iff \forall n_1, n_2(n_2$ is the E_y successor of $n_1 \to ((x)_{n_1}, (x)_{n_2}) \in Ult)$, and thus is Δ^1_2. Moreover, the shows that the section of A along y is also Δ^1_2. $\square$

Thus x codes a z-mouse is a $\Pi^1_2(z)$ property.

8 Problem 10.8

Let $x^\sharp = (J_\alpha[x], U)$, then $\omega^\omega \cap x^\sharp = \omega^\omega \cap L[x]$ by a condensation argument as in Problem 10.10. Hence by Corollary 7.21 the conclusion follows.

9 Problem 10.9

Claim 4 (Claim 10.9.1)**.** *For $J_\alpha[x], J_\beta[x]$, if $j : J_\alpha[x] \to J_\beta[x]$ is an elementary embedding which has critical point $\gamma < |\alpha|$, then $x^\sharp$ exists.*

Proof of claim: Let U be the ultrafilter defined on γ with j. Since $\gamma < |\alpha|$, $J_\alpha[x]$ and $L[x]$ agrees on $\mathcal{P}(\gamma)$ and hence $L[x]$ also thinks U is a γ complete ultrafilter on γ. It suffice to show that $Ult(L[x], U)$ is well-founded. Thus $Ult(L[x], U) = L[x]$ and the ultrapower map is an non trivial elementary embedding from $L[x]$ to itself.

Suppose for contradiction that $\dots [f_1] \in [f_0]$, let $J_\theta[x]$ be such that $f_n \in J_\theta[x]$. Take:

$$\pi : J_\delta[x] \cong h_{J_\theta[x]}(\gamma \cup \{f_n \mid n \in \omega\}) \prec J_\theta[x]$$

Then we assume $\pi(g_n) = f_n$ and $\delta < \alpha$ since $|J_\delta[x]| = \gamma < \alpha$. Thus $g_n \in J_\delta[x] \subseteq J_\alpha[x]$, since π is elementary and is constant on γ, we have that $\{\xi \mid$

$g_n(\xi) \in g_m(\xi)\} \in U$ iff $\{\xi \mid f_n(\xi) \in f_m(\xi)\} \in U$. This means that $[g_0], [g_1] \ldots$ would be an ill-founded chain in $Ult(J_\alpha[x], U)$, but this model embeds into $J_\beta[x]$, a contradiction. $\square$

Claim 5 (Claim 10.9.2). *Let κ be ω_1-Erdos, then $\kappa \to [\omega_1]^{<\omega}_{2^\omega}$.*

See for instance Jech Lemma 17.29$\square$

(i) Now for the ω_1 Erdos cardinal κ, we consider the model $J_\kappa[x]$. Define the map $F : [\omega]^{<\omega} \to 2^\omega$ by the following, for $n, m \in \omega$, $\lambda_1 < \cdots < \lambda_n < \kappa$:

$$F(\lambda_1 \ldots \lambda_n) = \{n \mid J_\kappa[x] \models \varphi_n(\lambda_1 \ldots \lambda_n)\}$$

Then by the Claim 10.9.2 we obtain there is $X \subseteq \kappa$ of size ω_1 s.t. X is a set of indiscernibles for $J_\kappa[x]$, i.e. for any $\lambda_{i_1} < \cdots < \lambda_{i_n} \in X$ and $\lambda_{j_1} < \cdots < \lambda_{j_n} \in X$ and any φ,

$$J_\kappa[x] \models \varphi(\lambda_{i_1} \ldots \lambda_{i_n}) \iff J_\kappa[x] \models \varphi(\lambda_{j_1} \ldots \lambda_{j_n})$$

We consider the model

$$\pi : J_\alpha[x] \cong h_{J_\kappa[x]}(X) \prec J_\kappa[x]$$

We write $\pi(\xi_\alpha) = \lambda_\alpha$ for . Then in $J_\alpha[X]$, every $a \in J_\alpha[x]$ is of the form $h_{J_\alpha[x]}(n, \xi_{\alpha_1} \ldots \xi_{\alpha_n})$ for some n and ξ_{α_i} and $\{\xi_\alpha \mid \alpha < \omega_1\}$ is indiscernibles for $J_\alpha[x]$. Then for arbitrary $e : \omega_1 \to \omega_1$ that is order preserving, it induces an elementary embedding $J_\alpha[x] \to J_\alpha[x]$ by the following map:

$$\pi_e : h_{J_\alpha[x]}(n, \xi_{\alpha_1} \ldots \xi_{\alpha_n}) \mapsto h_{J_\alpha[x]}(n, \xi_{e(\alpha_1)} \ldots \xi_{e(\alpha_n)})$$

Moreover, the first ordinal moved will be less than $|\alpha|$ since it is countable, while $\alpha \geq \omega_1$ as $|h_{J_\kappa[x]}(X)| \geq \omega_1$. By Claim 10.9.1 $x^\sharp$ exists. $\square$

10 Problem 10.10

(a) We first show that for all ordinal $\delta \in J_\alpha[x]$, we have $\pi_U^{J_\alpha[x]}(\delta) = \pi_U^{L[x]}(\delta)$. It suffice to show that $\delta^\kappa \cap J_\alpha[x] = \delta^\kappa \cap L[x]$.

$\subseteq$ is obvious. For the other side, if $f \in \delta^\kappa \cap L[X]$, we take γ large enough s.t. $f \in J_\gamma[x]$ and:

$$\pi : (J_\beta[x], x) \cong (Hull_{L[x]}(TC(f)), x) \prec_{\Sigma_1} (L[x], x)$$

where $|J_\beta[x]| \leq max\{\delta, \kappa\} \leq \kappa^{+L[x]} = \alpha$. Hence $\beta \leq \alpha$ and thus $f \in J_\alpha[x]$.

Note: Here the condensation always applies as $x \subseteq \omega$, hence the structures are always elementary w.r.t. the predicate.

Now given $\pi_U^{J_\alpha[x]}(\delta) = \pi_U^{L[x]}(\delta)$ for all ordinal $\delta \in J_\alpha[x]$, we show that $\pi_U^{J_\alpha[x]} = \pi_U^{L[x]}|_{J_\alpha[x]}$. For arbitrary $a \in J_\alpha[x]$, we have $\delta < \kappa^{+L[x]}$ s.t. $L[x] \models rank_{<_{L[x]}}(a) = \delta$. By Σ_1 elementariness of ultrapower embedding, $Ult(L[x], U) \models$

$rank_{<_{L[x]}}(\pi_U^{L[x]}(a)) = \pi_U^{L[x]}(\delta)$. Hence $L[x] \models rank_{<_{L[x]}}(\pi_U^{L[x]}(a)) = \pi_U^{L[x]}(\delta) = \pi_U^{J_\alpha[x]}(\delta)$. This shows that $\pi_U^{L[x]}(a) = \pi_U^{J_\alpha[x]}(a)$.

(b) As for $f : \kappa \to J_\alpha[x]$ that is $\in J_\alpha[x]$, $\pi_U^{\mathcal{M}}(f)(\kappa) = \pi_U^{L[x]}(f)(\kappa)$ by (a). This shows that $Ult_0(\mathcal{M}) \subseteq Ult(L[x], U)$. Hence $Ult_0(\mathcal{M})$ is transitive.

$J_{\alpha'}[x] = \{\pi_U^{\mathcal{M}}(f)(\kappa) \mid f \in J_\alpha[x]^\kappa \cap J_\alpha[x]\} = \{\pi_U^{L[x]}(f)(\kappa) \mid f \in J_\alpha[x]^\kappa \cap L[x]\} = \pi_U^{L[x]}(J_\alpha[x])$. For the last equation, the $\subseteq$ side is easy. For the other side, if $a \in \pi_U^{L[x]}(J_\alpha[x])$, assume $a = \pi_U^{L[x]}(f)(\kappa)$ where $f \in L[x]^\kappa \cap L[x]$, we can alter f to g s.t. $g \in J_\alpha[x]^\kappa \cap L[x]$ and $a = \pi_U^{L[x]}(f)(\kappa) = \pi_U^{L[x]}(g)(\kappa)$.

(c) by induction.

(d) For $\alpha < \pi(\xi)$, we show that there is $\eta < \xi$ s.t. $\alpha < \pi(\eta)$, which concludes the proof. Let $\alpha = [f]_U \in Ult(L[x], U)$, we may assume $f : \kappa \to \xi$. But as $cf(\xi) > cf(\kappa)$, $sup f = \delta < \xi$.Hence $\alpha \leq \pi(\delta) < \pi(\delta + 1)$.

By induction we show that $\pi_{0,\alpha}(\xi) = \xi$ and $cf(\xi) > (2^{Card(\kappa)^+})$, the successor case: since $|\kappa_{\alpha+1}| < 2^{Card(\kappa_\alpha)^+}$, we have by the above conclusion $\pi_{0,\alpha+1}(\xi) = \pi_{0,\alpha}(\xi) = \xi$.

For the limit case since ξ is a limit and $\gamma \leq 2^{Card(\kappa_\alpha)^+}$, we have $cf(\xi) > (2^{Card(\kappa)^+})$. $\pi_{0,\gamma}(\xi) = \xi$ since if $\pi_{0,\gamma}(\alpha) \leq \bigcup_{\beta<\gamma} m_\beta$ where m_β is taking β many power for $|\alpha|$, e.g. $m_1 = 2^{|\alpha|}$, $m_1 = 2^{2^{|\alpha|}}$ ... Which is still less than ξ since ξ is strong limit. □

11 Problem 10.11

Let $\mathcal{M}_0 = x^\sharp$ and $I = \{\kappa_\alpha \mid \alpha < \omega_1\}$ be the set of countable silver indiscernibles. We aim to show that for each $X \in \mathcal{P}(\omega_1) \cap L[x]$, there is α s.t. either

$$\{\kappa_\beta \mid \alpha < \beta\} \subseteq X \text{ or } \{\kappa_\beta \mid \alpha < \beta\} \subseteq \omega_1 \setminus X$$

And the conclusion follows from 10.2 (b).

Take the ω_1 itreration of $x^\sharp$, we have that

$$X \in \mathcal{P}(\omega_1) \cap L[x] \Rightarrow X \in \mathcal{M}_{\omega_1}$$

since ω_1 is the largest cardinal in $\mathcal{M}_{\omega_1}$ and thus we know that $\mathcal{P}(\omega_1) \cap L[x] = \mathcal{P}(\omega_1) \cap \mathcal{M}_{\omega_1}$ by the argument in 10.10 (a).

Hence, $X = \pi_{\alpha,\omega_1}(Y)$ for some $\alpha < \omega_1$ and $Y \in \mathcal{P}(\kappa_\alpha) \cap \mathcal{M}_\alpha$. We show that if $X \in U_\alpha$ then $\{\kappa_\beta \mid \alpha < \beta\} \subseteq X$ and the other case is similar.

The proof mirrors the argument in Lemma 10.9. For arbitrary $\beta > \alpha$, we consider the function for all $\xi \in \omega_1$,

$$\varphi(\xi) = \begin{cases} \xi & \text{if } \xi \leq \alpha \\ \xi + \beta - (\alpha + 1) & \text{if } \xi > \alpha \end{cases}$$

By the Shift lemma, we have

$$Y \in U_\alpha \iff \kappa_{\alpha+1} \in \pi_{\alpha\alpha+1}(Y) \iff \pi_{0\alpha+1}(\kappa) \in X \iff \pi_{0\beta}(\kappa) = \pi^\varphi_{\omega_1\omega_1}(\pi_{0\alpha+1}(\kappa)) \in \pi^\varphi_{\omega_1\omega_1}(X) = X$$

□

12 Problem 10.12(Not yet done)

We say a real x codes a structure $(M, \in, A)$ iff for $o(x) : n \mapsto 2n+1$ and $e(x) : n \mapsto 2n$ $\pi(\omega, E_{o(x)}, A_{e(x)}) \cong (M, \in, A)$ where $E_{o(x)}$ denotes the standard coding of $o(x)$ as a well-founded relation and $n \in A_{e(x)} \iff e(x)(n) \neq 0$. π is the transitive collapse.

Claim 6 (Claim 10.12.1)**.** *The following relation is Σ^1_2:*

$$(x, y) \in A \iff x \text{ codes a premouse and its iteration up to } ||y||.$$

In the sense that for all n, $(x)_n : m \mapsto x(\Gamma(n, m))$ codes a premouse and

$$n_1 E_y n_2 \iff (x)_{n_2} \text{ codes an ultrapower of } (x)_{n_1}$$

Proof. First we show that x codes a premouse is Π^1_1. x codes a premouse iff $o(x) \in \mathsf{WF}$ and $(\omega, E_{o(x)}) \models ZFC^- + V = L +$there is a largest cardinal and say n_κ is the largest cardinal in $(\omega, E_{o(x)}$ $(\omega, E_{o(x)}, U_{e(x)}) \models U_{e(x)}$ is a non-trivial normal $<$ κ complete ultrafilter on κ.

$(\omega, E_{o(x)}) \models ZFC^- + V = L +$ there is a largest cardinal is arithmetical since the relation $(\omega, E_{o(x)}) \models \varphi(n_1 \dots n_m)$ is arithmetical. To analyze $(\omega, E_{o(x)}, U_{e(x)}) \models U_{e(x)}$ is a non-trivial normal $< \kappa$ complete ultrafilter on κ, for instance $(\omega, E_{o(x)}, U_{e(x)}) \models U_{e(x)}$ is $< \kappa$ complete. iff

$$\forall n(\pi_x(n) \in \pi_x(n_\kappa) \to \forall X \in [\omega]^\omega (\forall m \in X(m \in U_{e(x)} \wedge \pi_x(m) \in \pi_x(n)) \to \exists l \in U_{e(x)}(\bigcap \pi_x[X] = \pi_x(l)))$$

Where π_x is the Mostowski collapse. This is a Π^1_1 property, for the Mostowski collapse part see for instance Jech Proof of Lemma 25.25.

Next consider the following relation:

$$(x, y) \in Ult \iff y, x \text{ both code premice and } y \text{ codes the ultrapower of } x$$

First we notice the ultrapower equivalence relation for x is given by, for n_1, n_2 that are functions with domain ω in $(\omega, E_{o(x)})$ collapsed,

$$n_1 \equiv n_2 \iff \exists m(m \in U_{e(x)} \wedge \forall l(\pi(l) \in \pi(m) \to \pi(n_1)(\pi(l)) \in \pi(n_2)(\pi(l)))) \tag{*}$$

We have y codes the ultrapower of x iff there is $\equiv_x \subseteq \omega^2, f \in \omega^\omega$ s.t. $\equiv_x$ is an equivalent relation satisfying $*$, f respects $\equiv_x$ and is bijective $\omega \to \{n \in \omega \mid \pi_x(n) \text{ is a function with domain } \omega\}/\equiv_x$,

$$n_1 E_{e(y)} n_2 \iff \exists m(m \in U_{e(x)} \wedge \forall l(\pi(l) \in \pi(m) \to \pi(n_1)(\pi(l)) \in \pi(n_2)(\pi(l))))$$

and

$$n_1 \in U_{e(y)} \iff m(m \in U_{e(x)} \wedge \forall l(\pi(l) \in \pi(m) \to l \in U_{e(x)}))$$

And hence Ult is a Σ^1_1 relation.

Hence, let $(x)_n : m \mapsto x(\Gamma(n, m))$, $(x, y) \in A \iff \forall n_1, n_2(n_2$ is the E_y successor of $n_1 \to ((x)_{n_1}, (x)_{n_2}) \in Ult)$, and thus is Δ^1_2. Moreover, the shows that the section of A along y is also Δ^1_2. $\square$

Thus given the claim, we can already prove that for any $\beta < \omega_1^L$ there is $\beta' > \beta$, α s.t. the there is premouse $(J_\alpha, \in, U) \in L$ and the putative iteration of length $\beta' + 1$ whose last model is ill-founded. For each $\beta < \omega_1^L$, pick $y \in \omega^\omega \cap L$ s.t. $||y|| = \beta$, since by Claim the section $A_y = \{x \mid (x, y) \in A\}$ is Δ^1_2. $0^\sharp$ witnesses its non-emptiness in V, by Shoenfield absoluteness Cor 7.21 A_y is non-empty in L. say $x \in A_y$. But the minimal premouse coded in x cannot be iterable since otherwise $0^\sharp \in L$, which is nonsense, hence some step of the iteration greater than β must fail.

Next we show that for any $\beta < \omega_1^L$ there is α s.t. there is premouse $(J_\alpha, \in, U) \in L$ and the putative iteration of length $\beta + 1$ whose last model is ill-founded. Pick the L least element $J_\alpha, \in, U$ s.t. there is $x \in A_y$ s.t. $(x)_{n^*}$ codes $J_\alpha, \in, U$ where n^* is the least element in the order coded by y. We build a tree of attempts to find elementary embeddings from the $\beta + 1$ th iterate of $J_\alpha, \in, U$ into some large enough model.

13 Problem 10.13

We use the fact that $Col(\omega, <\kappa) = \prod^{fin}_{\lambda<\kappa} Col(\omega, \lambda)$. Let $(\kappa_\alpha, \alpha < \omega_1)$ be the countable Silver indiscernibles, as discussed in problem 10.11. Let $(\mathcal{M}_\alpha, \pi_{\alpha\beta}, \alpha \leq \beta < \omega_1)$ be the iteration of $x^\sharp$ up to ω_1.

We prove by induction that $\alpha < \omega_1$, there is $G_\alpha \in V$ s.t. that is $Col(\omega, <\kappa_\alpha)$-generic over $L[x]$. And if $\alpha < \beta$, then G_α, G_β are consistent in the sense that if $p \in Col(\omega, <\kappa_\beta)$ has support contained κ_α, it holds true that $p \in G_\alpha \iff p \in G_\beta$

The base case: κ_0 is countable in V and all dense sets in $Col(\omega, <\kappa_0)$ that is in $L[x]$ is already contained in $\mathcal{M}_0$, thus is countable. By the generic filter theorem there is $Col(\omega, <\kappa_0)$ generic G_0 over $L[x]$.

The successor case: $Col(\omega, <\kappa_{\alpha+1}) = Col(\omega, <\kappa_\alpha) \times \prod^{fin}_{\kappa_\alpha \leq \lambda < \kappa_{\alpha+1}} Col(\omega, \lambda)$, by a similar argument to the base case, we have $\prod^{fin}_{\kappa_\alpha \leq \lambda < \kappa_{\alpha+1}} Col(\omega, \lambda)$ generic H over $L[x]$ in V. Let $G_{\alpha+1} = G_\alpha \times H$ and by Lemma 6.65 this satisfies the requirement.

The limit case: Define $q \in G_\gamma \iff$ the support of q is contained in $\kappa_\alpha, q \in G_\alpha$. We have that G_γ is generic as suppose A is an antichain of $Col(\omega, <\kappa_\gamma)$, by the fact that ω_γ is inaccessible in $L[x]$, by Lemma 6.44 $L[x]$ thinks $|A| < \kappa_\gamma$, thus for some $\alpha < \gamma$, and thus A can be considered as an antichain in $Col(\omega, <\kappa_\alpha)$. By the consistency of G_γ w.r.t. G_α, it intersects A.

Next, we consider the filter G on $Col(\omega, <\omega_1)$ defined by

$$p \in G \iff \text{suppose the support of } q \text{ is contained in } \kappa_\alpha, \text{then } p \in G_\alpha$$

By the exact same argument as the limit case, we have that G is generic. □

14 Problem 10.14(Not yet done)

Claim 7 (Claim 10.14.1)**.** *A remarkable cardinal is inaccessible.*

Proof. To show it is regular, pick arbitrary function $f : \delta \to \kappa$ where $\delta < \kappa$. We pick $\alpha > \kappa$ s.t. f exists in V_α. In the generic extension, there is $\sigma : V_\beta \to V_\alpha$ with critical point $\mu, \sigma(\mu) = \kappa$. Since $V_\beta \models \mu$ is inaccessible, $V_\alpha \models \kappa$ is inaccessible, contradicting the existence of f. The proof for strong limit is similar. □

Claim 8 (Claim 10.14.2)**.** *For regular κ, κ-c.c. forcing preserves stationary in* $[\lambda]^{<\kappa}$

Proof. For arbitrary $\dot{C}$ s.t. $p \Vdash \dot{C}$ is club, we pick τ s.t. $p \Vdash \tau \in \dot{C}$. We subsequently pick nice name τ_n, $X_n \in V$ s.t. $p \Vdash \tau_n \in \dot{C}$, $p \Vdash \check{X}_n \subseteq \tau_n$, $X_n \in [\lambda]^{<\kappa}$ and $ran(\tau_n) \subseteq X_n \in [\lambda]^{<\kappa}$. The final requirement is doable by κ-c.c. forcing. The sequence is definable in V, and thus we take $\bigcup_n X_n$, which is in V and p forces it to be in $\dot{C}$. This argument shows that the limit point of $\dot{C}_G$ is a club set in V. This entails that the forcing preserves stationary in $[\lambda]^{<\kappa}$. □

(a) For arbitrary α, pick $\sigma : V_\beta \to V_\alpha$ in $V[G]$ s.t. $crit(\sigma) = \mu$ and $\sigma(\mu) = \kappa$. The idea is to show that in, $S = \{X \in [V_\beta]^\omega \mid X \prec V_\beta, X \cap \mu \in \mu, \exists \beta' X \cong V_{\beta'}\}$ is stationary and lift this statement via σ, then use the stationary preservation.

Now by the proof of Madgidor's characterisation of supercompact cardinal problem 4.29 and problem 4.30, we have that there is a normal V-ultrafilter U on $([V_\beta]^{<\mu})^V$ generated by

$$X \in U \iff \sigma[V_\beta] \in \sigma(X)$$

Hence we obtain that the set S is in U as of course $\sigma[V_\beta] \in \sigma(S) = \{X \in [V_\alpha]^\omega \mid X \prec V_\alpha, X \cap \kappa \in \kappa, \exists \beta X \cong V_\beta\}$. This means that S intersects all V club in $[V_\beta]^{<\mu}$. Lift this up and hence $\{X \in [V_\alpha]^{<\kappa} \mid X \prec V_\alpha, X \cap \kappa \in \kappa, \exists \beta' X \cong V_{\beta'}\}$ intersects all V club in $[V_\alpha]^{<\kappa}$. Since $Col(\omega, < \kappa)$ is κ- c.c., stationary set of $[V_\alpha]^{<\kappa}$ is preserved and hence it is stationary in $[V_\alpha]^\omega$ in V[G]. □

(b) If $0^\sharp$ exists, for Silver indiscernible

(c) If κ is a remarkable cardinal in V, then for arbitrary α there is in $V[G]$ an elementary embedding $\sigma : V_\beta \to V_\alpha$ with $crit(\sigma) = \mu$ and $\sigma(\mu) = \kappa$. It suffice to argue that some embedding $\sigma' : (V_\beta)^L \to (V_\alpha)^L$ exists in $L[G]$.

Claim 10.14.3 If $j : M \to N$ is an elementary embedding in V and M is countable, then for any transitive model H that knows enough ZFC so that well-foundedness is absolute s.t. $M, N \in H$ and M is countable in H, then there is elementary embedding $j' : M \to N$ in H

Proof. Define the following tree of partial elementary map: Fix an enumeration $m_i, i \in \omega$ of elements in M. $p \in T$ is a partial elementary map from M to N with finite domain. $p_1 \leq p_2$ if p_1 is an end extension of p_2. Then we have

$$T \text{ is ill-founded} \iff \exists j : M \to N \text{ elementary}$$

By the assumption that ill-foundedness is absolute between M and N, we obtain the desired result.

Now $\sigma|_{(V_\beta)^L}$ is an elementary map from countable structure V_β^L to V_α^L. V_β^L is still countable in $L[G]$ as $\beta < \kappa$. By the claim 10.14.3 we thus obtain that such an elementary map exists in $L[G]$. □

15 Problem 10.16

Claim 9 (Claim 10.16.1)**.** *If V is closed under sharps then for all $X \subseteq Ord$ and $X \in V$, $X^\sharp$ exists.*

This is essentially Jech Exercise 18.2

Proof of Claim: We take $\kappa > |X|$ and take H generic over $Col(\omega, \kappa)$, then X is countable in $V[G]$ and hence $X^\sharp$ exists. But the statement that $X^\sharp$ exists is equivalent to the statement that $L[X]$ has a proper class of Silver indiscernibles. Take $\{\kappa_\alpha \mid \alpha > \beta\}$ be the class of cardinals in $V[G]$ greater than $|\mathbb{P}|$, they are also cardinals in V. Notice $L \models \varphi(\kappa_{\alpha_1} \dots \kappa_{\alpha_n})$ iff $L \models \varphi(\kappa_{\alpha'_1} \dots \kappa_{\alpha'_n})$, is absolute for $V[G]$ and V and hence $\{\kappa_\alpha \mid \alpha > \beta\}$ is a proper class of Silver indiscernibles in V. This shows that $X^\sharp$ exists in V. □

Let G be generic over $\mathbb{P}$. Let A be such that $V[G] \models x \in A \iff \exists y \varphi(x, y, z)$ for some φ that is $\Sigma^1_2(z)$. We take $X \in V$ s.t. $\mathbb{P} \in X^\sharp = (J_\alpha[X], \in, U)$ and $\mathbb{P} \in J_\kappa[X]$, where κ is the largest cardinal in $J_\alpha[x]$. Say

$$p \Vdash \varphi(\tau_1, \tau_2, z)$$

We take $\pi : N \to X^\sharp$ elementary be s.t. N countable transitive, $\pi(q) = p$ and $\pi(\mathbb{Q}) = \mathbb{Q}$, $\pi(\sigma_1) = \tau_1$ and $\pi(\sigma_2) = \tau_2$. Hence

$$N \models q \Vdash \varphi(\tau_1, \tau_2, z)$$

By assumption that $\mathbb{P} \in J_\kappa[X]$, $\mathbb{Q}$ is contained in the part that is not moved by iteration.

By Claim 10.1.1, we may iterate N up to ω_1, call it N_{ω_1}, then $\omega_1 \subseteq N_{\omega_1}$ and $q, \mathbb{Q}, \sigma_1, \sigma_2$ is unchanged under the iteration. Notice that the subsets of $\mathbb{Q}$ in N_{ω_1} are already appearing in N, which is countable. Hence, there is a $\mathbb{Q}$ generic $g \in V$ over N_{ω_1}, and we have

$$N_{\omega_1}[g] \models \varphi(\sigma_{1g}, \sigma_{2g}, z)$$

As $\sigma_{1g}, \sigma_{2g} \in V$ by Shoenfield absoluteness,

$$V \models \exists x, y \varphi(x, y, z)$$

Concluding the proof. □

16 Problem 10.17

Right to Left: Assume for contradiction that $\dots \in [a_2, f_2] \in [a_1, f_1]$.

By Lemma 10.64, we may have

$$h_{Ult_0(V;E)}(\{[a_n, f_n] \mid n \in \omega\}) \prec_{\Sigma_0} Ult_0(V; E)$$

and Σ_0 elementary map

$$\varphi : h_{Ult_0(V;E)}(\{[a_n, f_n] \mid n \in \omega\}) \to V$$

This leads to the non well-foundedness of V, a contradiction.

Left to Right: Assume $Ult(V, E) \cong M$ is well founded. Let j_E be the extender embedding. Consider the tree

$$U := \{s \mid \exists k \in \omega(s : \bigcup_{i \le k} a_i \to \kappa \wedge s \text{ is order preserving } \wedge \forall i \le k, s[a_i] \in X_i)\}$$

where the order is $s_1 \prec s_2 \iff s_2 \subset s_1$. The inverse of $j|_{\bigcup_{i\in\omega} a_i}$ witnesses that $j(U)$ is not a well-founded tree as for each k,

$$(j|_{\bigcup_{i\le k} j(a_i)})^{-1} : \bigcup_{i\le k} j(a_i) \to j(\kappa) \text{ is order preserving and } \forall i \le k (j|_{\bigcup_{i\le k} j(a_i)})^{-1}(j(a_i)) = a_i \in X_i$$

By the absoluteness of well-foundedness, $j(U)$ is not well-founded in M, and thus U is not well-founded in V by elementarity. This gives the desired map. □

17 Problem 10.18

(1) For $cf(\alpha) < \kappa$, the argument is exactly the same as Lemma 4.52 (c).

For $cf(\alpha) > \kappa$, let $\beta_\gamma \to \alpha, \gamma \to cf(\alpha)$. Of course $\bigcup_{\gamma\to cf(\alpha)} \pi_E\beta_\gamma \le \pi_E\alpha$. For arbitrary $\xi < \pi_E\alpha$, $\xi = [a, f]$ for some $a \in [\nu]^{<\omega}, f : [\mu_a]^{|a|} \to \alpha$. By the fact that E is a short extender, we have $|\mu_a| \le \kappa$, then f has to be bounded by some $\pi_E\beta_\gamma$, hence $\xi < \pi_E\beta_\gamma$.

(2) Pick a cofinal sequence $\beta_\gamma \to \lambda, \gamma \to cf(\lambda)$. By (1) we thus have $sup_{\gamma\to cf(\lambda)}\pi_E\beta_\gamma = \pi_E\lambda$. We show that $\pi_E\beta_\gamma < \lambda$ and this concludes the proof. Since each $\xi < \pi_E\beta_\gamma$ is of the form $\xi = [a, f]$ for some $a \in [\nu]^{<\omega}, f : [\mu_a]^{|a|} \to \beta_\gamma$, and as E is a short extender, we have $|\mu_a| \le \kappa$. We have that $|\pi_E\beta_\gamma| \le |\nu \times \beta_\gamma^\kappa| < \lambda$. This concludes the proof. □

18 Problem 10.19

Ultrafilter Property: For $\alpha < \kappa$, $(Y_i \mid i < \alpha) \in V[G] \cap E_a^{*\alpha}$. Let $p \in G$. Let A_i be a maximal antichain, definable in V, of elements $p \le q$ s.t. $\exists X_{q,i}, q \Vdash X_{q,i} \subseteq \dot{Y}_\alpha$. Then since $|\mathbb{P}| < \kappa$, $|A_i| < \kappa$ and hence

$$\bigcap_{i<\alpha} Y_i \supseteq \bigcap_{i<\alpha, q\in A_i} X_{i,q} \in E_a$$

To Checking that this is an ultrafilter, upward closedness is easy. For $Y \in [\mu_a]^{|a|} \cap V[G]$, consider the following sets, definable in V: $D_u := \{p \mid p \Vdash u \in \dot{Y}\}, F_u = \{u \mid p \Vdash u \notin \dot{Y}\}$. By the fact that κ is inaccessible in V, $\{D_u \mid u \in [\mu_a]^{|a|}\} \subseteq \mathcal{P}(|\mathbb{P}|)$ and thus is of cardinality less than κ. Hence there is $X \in E_a$ s.t. $D_u = D_{u'}, F_u = F_{u'}$ for all $u, u' \in X$. Now $D_u \cup F_u$ is dense in $\mathbb{P}$ and hence G intersects elements of it, but it can't intersect element from both D_u and F_u. If $G \cap D_u$ is not empty, then $Y \supseteq X$ and hence $Y \in E_a^*$, the other side is similar.

Remark:If we naively take $(X_i \mid i < \alpha)$ subsets of $(Y_i \mid i < \alpha)$, the sequence might not be in V. We circumvent this by considering all possible subsets of Y_i in U.

We notice that $(*)$ for all ordinal μ, $[\mu]^{|a|} \in E_a \iff [\mu]^{|a|} \in E_a^*$ and hence μ_a is the smallest μ s.t. $[\mu]^{|a|} \in E_a \iff [\mu]^{|a|} \in E_a^*$.

Coherence: For $Y \in E_a^*$ and $b \supseteq a$, we have that $Y \supseteq X$ for some $X \in E_a$ and thus $X^{ab} \in E_b$. Since $X^{ab} \subseteq Y^{ab}$, $Y^{ab} \in E_b^*$.

The other side follows from the fact that if $Y \notin E_a^*$, then $[\mu_a]^|a| - Y \in E_a^*$ but $[\mu_a]^|a| - Y^{ab} \cap Y^{ab} = \emptyset$.

Uniformity follows from $(*)$.

Normality: Take $f : [\mu_a]^{|a|} \to \mu_a$ with $f \in V[G]$,

$$p \Vdash \exists X, X \subseteq \{u \mid \dot{f}(u) < max(u)\}$$

Case 1: $\mu_a < \kappa$. We have $p \Vdash \bigcup_{g:[\mu_a]^{|a|}\to\mu_a} \{u \mid g(u) = \dot{f}(u)\} = [\mu_a]^{|a|}$ since $p \Vdash \exists g : [\mu_a]^{|a|} \to \mu_a, g(u) = \dot{f}(u)$ for all u. Since there are at only $|\mu_a|^{\mu_a}$ many such functions $g : [\mu_a]^{|a|} \to \mu_a$, $p \Vdash \exists g, \{u \mid g(u) = \dot{f}(u)\} \in E_a$. Since we have $\{u \mid g(u) < max(u)\} \in E_a$, we obtain by normality that there is β s.t. $\{u \mid g^{a,a\cup\{\beta\}}(u) < max(u)\} \in E_{a\cup\{\beta\}}$. The conclusion follows as

$$\{u \mid g^{a,a\cup\{\beta\}}(u) < max(u)\} \cap \{u \mid g^{a,a\cup\{\beta\}}(u) = \dot{f}^{a,a\cup\{\beta\}}(u)\} \subseteq \{u \mid \dot{f}^{a,a\cup\{\beta\}}(u) < max(u)\}$$

Case 2: $\mu_a \geq \kappa$. Idea: $< \kappa$ forcing should preserve stationary sets for $[\mu_a]^{<\omega}$, $\mu_a > \kappa$.

Claim 10 (Claim 10.19.1)**.** *For κ regular, forcing of size $< \kappa$ preserves stationary sets on μ where $\mu \geq \kappa$.*

Proof of Claim. It suffice to show that if $p \Vdash \dot{C}$ is a club, then $p \Vdash \{\alpha \mid \alpha \in \dot{C}\}$ is a club. That $p \Vdash \{\alpha \mid \alpha \in \dot{C}\}$ is closed is easy. For α_0 s.t. $p \Vdash \alpha_0 \in \dot{C}$, we have a name $\dot{\gamma_0}$ for an ordinal s.t. $p \Vdash \alpha_0 < \dot{\gamma_0} \in \dot{C}$. Pick a maximal antichain A under p where $q \in A$ entails $q \Vdash \dot{\gamma_0} = \gamma_q$. By the fact that the forcing is of size $< \kappa$, we have $\alpha_1 = sup\{\gamma_q \mid q \in A\} < \kappa$ and thus

$$p \Vdash \exists \dot{\gamma_0} \in \dot{C}, \alpha_0 < \dot{\gamma_0} < \alpha_1$$

Repeat this process to find $\alpha_n, n \in \omega$ s.t.

$$p \Vdash \exists \dot{\gamma_n} \in \dot{C}, \alpha_n < \dot{\gamma_n} < \alpha_{n+1}$$

Of course, p thinks $\dot{\gamma_n}$ and α_n shares a limit, and by closedness of $\dot{C}$, $\lim_{n\in\omega} \alpha_n \in \{\alpha \mid \alpha \in \dot{C}\}$. $\square$

Remark: The argument actually works for κ c.c. posets, and is the argument used to show that c.c.c. forcing is proper.

Proof of case 2: Fix an enumeration $\Gamma : [\mu_a]^{|a|} \to \mu_a$ satisfying $\Gamma(u) > max(u)$. And consider the induced map $\Gamma : \mathcal{P}([\mu_a]^{|a|}) \to \mathcal{P}(\mu_a)$

Then, the normality of E_a is equivalent to saying that $\{\Gamma(X) \mid X \in E_a\}$ contains all closed sets in μ_a (normal as an ultrafilter on μ_a). Then the conclusion follows by Claim 10.19.1. $\square$

Next, we show that the two large cardinals are preserved under small forcing.

Claim 11 (Claim 10.19.2)**.** *For forcing $\mathbb{P}$ of size $< \kappa$, the extender elementary embedding $\pi_E : V \to Ult(V, E)$ extends to elementary $\pi_{E^*} : V[G] \to Ult(V[G], E^*)$, satisfying*

$$\pi_{E^*} : \tau_G \mapsto \pi_E(\tau)_G = [\emptyset, c_{\tau_G}]_{E^*}$$

And consequently, $Ult(V[G], E^*) = Ult(V, E)[G]$.

Proof. To show that π_{E^*} is elementary, it suffice to show that $\pi_E(\tau)_G = ([\emptyset, c_\tau]_E)_G = [\emptyset, c_{\tau_G}]_{E^*}$. We prove by an induction on the well-foundedness of elements in $Ult(V[G], E^*)$ that

$$[a, \dot{f}_G]_{E^*} = ([a, \dot{f}]_E)_G$$

Notice as the domain $[\mu_a]^{|a|}$ is absolute, here we can slightly abuse the notation, by $\dot{f}$ refers to both the function from $[\mu_a]^{|a|}$ to names and a name of the function from $[\mu_a]^{|a|}$ to elements in $V[G]$.

Say for $[b, \dot{g}_G]_{E^*} \in [a, \dot{f}_G]_{E^*}$, by IH we have $[b, \dot{g}_G]_{E^*} = ([b, \dot{g}]_E)_g$.

$$\begin{aligned}
[b, \dot{g}_G]_{E^*} \in [a, \dot{f}_G]_{E^*} &= \exists X \in E_{a\cup b}, X \subseteq \{u \in [\mu_{a\cup b}]^{|a\cup b|} \mid \dot{g}_G^{b,a\cup b}(u) \in \dot{f}_G^{a,a\cup b}(u)\} \\
&\iff \exists X \in E_{a\cup b}, X \subseteq \{u \in [\mu_{a\cup b}]^{|a\cup b|} \mid \exists p \in G, p \Vdash \dot{g}^{b,a\cup b}(u) \in \dot{f}^{a,a\cup b}(u)\} \\
&\iff \exists p \in G, p \Vdash \{u \in [\mu_{a\cup b}]^{|a\cup b|} \mid \dot{g}^{b,a\cup b}(u) \in \dot{f}^{a,a\cup b}(u)\} \in E_{a\cup b} \qquad (*) \\
&\iff ([b, \dot{g}]_E)_G \in ([a, \dot{f}]_E)_G
\end{aligned}$$

Here the only non trivial step is $(*)$, where left to right is by the fact that $|G| < \kappa$ and $E_{a\cup b}$ is $< \kappa$ complete.$\square$

Proof of the preservation of large cardinals.

For κ a strong cardinal, we take arbitrary $\alpha \geq \kappa + 2$, and show that there is $j : V[G] \to M$ elementary with $crit(j) = \kappa$, $(V_\alpha)^{V[G]} \subseteq M$.

Let E be the κ, ν-extender obtained in Lemma 10.58 where $\alpha < \nu$. For $x \in (V_\alpha)^{V[G]}$, as $|\mathbb{P}| < \kappa$ we may assume $\mathbb{P} \in V_\kappa$, we have $\dot{x} \in V_\alpha$ thus in $Ult(V, U)$. By Claim 10.19.2 $Ult(V[G], E^*) = Ult(V, E)[G]$ computes the name correctly and thus $x = \dot{x}_G \in Ult(V[G], E^*)$. Hence $(V_\alpha)^{V[G]} \subseteq Ult(V, E^*)$. This concludes the proof that κ is strong in $V[G]$.

For κ a supercompact cardinal, similarly, let E be given by Lemma 10.61 and we aim to show that $Ult(V[G], E^*)^\lambda \subseteq Ult(V[G], E_*)$.

For $\{[a_i, \dot{f}_{i_G}]_{E^*} \mid i < \lambda\} \subseteq Ult(V[G], E^*)$, we have by Claim 10.19.2 that $\{([a_i, \dot{f}_i]_E)_G \mid i < \lambda\} \subseteq Ult(V, E)[G]$. This shows that $\{[a_i, \dot{f}_i]_E \mid i < \lambda\} \subseteq Ult(V, E)$. By λ closedness of $Ult(V, E)$ and pass it back to $Ult(V[G], E^*)$, we are done. $\square$

19 Problem 10.20

This is similar to preservation of strongness. Given δ Woodin in V, it suffice to show for $A \subseteq (V_\delta)^{V[G]}$ there is κ s.t. for any $\alpha < \kappa$ there is elementary $\pi : V[G] \to M$ s.t. $crit(\pi) = \kappa$, $(V_\alpha)^{V[G]} \subseteq M$ and $\pi(A) \cap (V_\alpha)^{V[G]} = A \cap (V_\alpha)^{V[G]}$.

Fix $A \subseteq (V_\delta)^{V[G]}$, $\mathbb{P}$ being small, we may assume in the similar fashion as in proof of preservation of strongness that $\dot{A}$, the name of A, to be a subset of V_δ. We apply Lemma 10.77 and pick κ satisfying the for all $\alpha < \kappa$ there is certified E s.t. $\pi_E : V \to M$ is elementary and $crit(\pi) = \kappa$, $V_\alpha \subseteq M$ and $\pi_E(\dot{A}) \cap V_\alpha = \dot{A} \cap V_\alpha$.

It suffice to show that for any $\alpha > |\mathbb{P}| + 2$, the corresponding E satisfies: $\pi_{E_*}(A) \cap (V_\alpha)^{V[G]} = A \cap (V_\alpha)^{V[G]}$. We have by claim 10.19.2,

$$\begin{aligned}
\tau_G \in (V_\alpha)^{V[G]} \cap A &\iff \exists p \in G, p \Vdash \tau \in \dot{A} \cap V_\alpha \\
&\iff \exists p \in G \exists B \text{ a maximal antichain under } p, B \times \{\tau\} \subseteq \dot{A} \cap V_\alpha \\
&\iff \exists p \in G \exists B \text{ a maximal antichain under } p, B \times \{\tau\} \subseteq \pi_E(\dot{A}) \cap V_\alpha \\
&\iff \exists p \in G, p \Vdash \tau \in \pi_E(\dot{A}) \cap V_\alpha \\
&\iff \tau_G \in (V_\alpha)^{V[G]} \cap (\pi_E(\dot{A}))_G = (V_\alpha)^{V[G]} \cap \pi_{E_*}(A)
\end{aligned}$$

This concludes the proof. □

20 Problem 10.21

Don't know how to argue via extenders, don't know how to show the derived extender is λ closed. We solve this probem by proving the equivalence of Problem 10.22 first, and using Problem 10.22,

Assume for contradiction that κ is not supercompact, then let $\lambda \geq \kappa$ be the least cardinal s.t. there is no ultrafilter U on $\mathcal{P}_\kappa(\lambda)$ witnessing κ is not λ supercompact. This is a first order property.

Pick α s.t. $V_\alpha^\lambda \subseteq V_\alpha$, any limit α s.t. $cf(\alpha) > \lambda$ would satisfy the property. Then by assumption there is $\mu < \beta < \kappa \leq \lambda < \alpha$ s.t. there is $\sigma : V_\beta \to V_\alpha$ elementary, with $crit(\sigma) = \mu$. Notice $V_\beta \models$ there is least δ s.t. there is no ultrafilter U on $\mathcal{P}_\kappa(\lambda)$ witnessing μ is not δ supercompact. Say δ is a witness of this statement, then by elementarity $\sigma(\delta) = \lambda$.

But deriving an ultrafilter U on $\mathcal{P}_\kappa(\lambda)$ from σ satisfying the conditions in 4.30, by Problem 10.22 the ultrafilter U in V_β witnesses the $\sigma^{-1}(\xi) = \delta$ supercompactness of μ in V_β. This is a contradiction.□

The use of ultrafilter turns the second-order property supercompactness to a first order property. cf. problem 10.23

21 Problem 10.22

For U a $< \kappa$ complete ultrafilter on $[\lambda]^{<\kappa}$, we show that the map $\pi_U : V \to Ult(V, U)$ witnesses that κ is λ supercompact. Here $Ult(V, U)$ is the ordinary ultrapower construction, adapted to U and $[\lambda]^{<\kappa}$. We omit the proof of Los theorem.

Step 1 $[id]_U = \pi_U[\lambda]$.

$\supseteq$ is by the first property and Los theorem: For $\alpha \in \lambda$, $\{a \in [\lambda]^{<\kappa} \mid \alpha \in a\} \in U$ and thus $\pi_U(\alpha) \in [id]_U$.

$\subseteq$ is by the second property, which is clearly the analogue of normality. Suppose $[f]_U \in [id]_U$, then we have $\{a \mid f(a) \in a\} \in U$, let $X_\alpha := \{a \mid f(a) = \alpha\}$. Suppose for contradiction that $X_\alpha \notin U$ for all $\alpha < \lambda$, then by the second property of the ultrafilter, there is $X \in U$ s.t. if $\alpha \in a \in X$, then $a \notin X_\alpha$, i.e. $f(a) \neq \alpha$. This set must be disjoint from $\{a \mid f(a) \in a\}$. This is a contradiction. Hence some $X_\alpha \in U$, entailing $[f]_U = \pi_U(\alpha) \in \pi_U[\lambda]$.

Step 2 $M^\lambda \subseteq M$.

Say $\{[f_\alpha] \mid \alpha < \lambda\} \subseteq M$. Let $g : [\lambda]^{<\kappa} \to V$ be s.t. $g(a)$ is a function $a \to V$, $g(a)(\alpha) = f_\alpha(a)$.

By Los theorem, $[g]_U$ is a function from $\pi_U[\lambda] \to Ult(V, U)$ and for every $\alpha \in \lambda$, $[g](\pi_U(\alpha)) = [f_\alpha]$. Hence $ran([g]) = \{[f_\alpha] \mid \alpha < \lambda\} \in M$.

Step 3 $crit(\pi_U) = \kappa$, $\lambda < \pi_U(\kappa)$.

$crit(\pi_U) \geq \kappa$ holds by $< \kappa$-completeness and the standard argument. We have $ot([id]_U) < \pi_U(\kappa)$ as $\{a \in [\lambda]^{<\kappa} \mid ot(a) < \kappa\} = [\lambda]^{<\kappa} \in U$. Moreover for arbitrary $\gamma < \kappa$, $\gamma \leq ot([id]_U)$ since

$$\{a \in [\lambda]^{<\kappa} \mid \gamma \leq ot(a)\} \supseteq \bigcap_{\alpha<\gamma} \{a \mid \alpha \in a\} \in U$$

By the first property and $< \kappa$ completeness. This shows that $\kappa \leq ot([id]_U) < \pi_U(\kappa)$ and thus $crit(\pi_U) = \kappa$.

Finally, $\lambda = ot(\pi_U[\lambda]) = ot([id]_U) < \pi_U(\kappa)$.

This concludes the proof. □

22 Problem 10.23

Claim 12 (Claim 10.23.1)**.** *Every subcompact cardinal is inaccessible.*

For arbitrary $A \subseteq V_\delta$, $A \subseteq H_{\delta^+}$. We find $\sigma : (H_{\lambda^+}, B) \to (H_{\delta^+}, A)$. For $crit(\sigma) = \mu$, σ satisfy for arbitrary $\beta < \lambda$, $V_\beta \subseteq H_{\delta^+}$ and $\sigma(B) \cap V_\beta = B \cap V_\beta$.

Work as the proof of Claim 10.79 in H_{λ^+} and extract an (μ, λ) extender E witnessing $\pi_E(B) \cap V_\beta = B \cap V_\beta$, the extender is in H_{λ^+}.

Pass the statement to H_{λ^+}, A and hence for all $\alpha < \delta$ there is $\sigma(\mu)$, there is $\sigma(E)$ a $(\sigma(\mu), \delta)$ extender on $\sigma(\mu)$, witnessing $A \cap V_\alpha = \pi_{\sigma(E)}(A) \cap V_\alpha$. □

23 Problem 10.24

Let $x := \bigcup\{s \mid \exists T \in G, s \text{ is the stem of } T\}$. For arbitrary $\delta \in [\kappa, \lambda]$ s.t. $cf(\delta) \geq \kappa$, we show that $\{\sup(a \cap \delta) \mid a \in x\}$ is unbounded in δ. And the conclusion $cf^{V[G]}(\delta) = \omega$ follows.

To that end, it suffice to show that

$$\{T \mid s \text{ is the stem of } T, \exists a \in s, \gamma \leq \sup a, a \cap [\gamma, \delta) \neq \emptyset\} \tag{*}$$

is dense. Such a satisfying the property above would have $\gamma \leq \sup(a \cap \delta) < \delta$ by $cf(\delta) \geq \kappa$.

For arbitrary T in the forcing with stem s, we notice that $\{a \mid s \frown a \in T\} \in U$ contains a set a^* that contains γ, that's because $\{a \mid \gamma \in a\} \in U$. Let T' be the subtree of T with stem $s \frown a^*$. This tree witnesses $(*)$ property.

Claim 13 (Claim 10.24.1)**.** *For all T and formula $\varphi(\tau_1 \ldots \tau_n)$, there is $T' \leq T$ with the same stem that decides the formula.*

Proof of Claim, For a tree T in the forcing with stem s, we use the notation (s, T) to make explicit its stem. For condition (s, T), define as in proof of Claim 10.7 $F : [X]^{<\omega} \to 3$ as follows:

$$F(s') = \begin{cases} 0 & \text{if there is no } T' \text{ s.t. } (s \cup s', T') \text{ decides } \varphi \\ 1 & \text{if there is } T' \text{ s.t. } (s \cup s', T') \text{ forces } \varphi \\ 2 & \text{if there is } T' \text{ s.t. } (s \cup s', T') \text{ forces } \neg\varphi \end{cases}$$

By definability of forcing, $F \in V$ and thus by Rowbottom's theorem there is $Y \in U$ s.t. for each $n \in \omega$, F is constant on $[Y]^n$.

Let (s, T') be the subtree of (s, T) defined recursively satisfying if $a \in succ_T(b)$, then

$$a \in T' \iff a \in succ_{T'}(b) \cap Y$$

(s, T') is the subtree of (s, T) slimed at every node by Y.

We show that (s, T') decides φ. If not, then there is $(b_1, T_1), (b_2, T_2) \leq (s, T')$ s.t. $(b_1, T_1) \Vdash \varphi$ while $(b_2, T_2) \Vdash \neg\varphi$. May assume $|b_1| = |b_2| = |a| + n$. But by design $b_1 \setminus a, b_2 \setminus a \in [Y]^n$ while $F(b_1 \setminus a) \neq F(b_2 \setminus a)$. This is a contradiction. □

Proof of $V_\kappa = V_\kappa^{V[G]}$: This is similar to Lemma 10.6. In the final step we take $q = (a, \bigcap_{\xi<\lambda} T_\xi)$, $\bigcap_{\xi<\lambda} T_\xi$ is still a valid tree by $< \kappa$ completeness of the ultrafilter. □

24 Problem 10.25

Assume for contradiction that $E_0 >_M E_1 >_M E_2 \ldots$. Consider the following iteration tree where $<_T := \{0\} \times \mathbb{N}^+$, $M_0 = V$, $M_{n+1} = Ult(M_0, E_n)$. As $E_n \in M_n$ by definition of Mitchell order, this is a one-level, infinitely branching iteration tree. This contradicts Theorem 10.74 as there is no infinite branch in this tree. □