Solutions to problems in Ch.11

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I currently wish to postpone Solidity and Box principle till later. As I haven't finished section 11.3, the solutions here are incomplete.

1 Problem 11.1

Step 1: By GCH, show that there is a class function $\varphi : Ord \to V$ satisfying $\varphi|_{\aleph_{\alpha+1}}$ is a surjection from $\aleph_{\alpha+1} \to \mathcal{P}(\aleph_\alpha)$.

This is because under GCH, by an inductive argument, for $\alpha \geq \omega$, $|V_\alpha| = \aleph_\alpha$.

Step 2: Code the function as a class of ordinals. Consider the following relation corresponding to the class function, $R(\alpha, \beta) \iff \beta \in \varphi(\alpha)$. Let Γ be the canonical pairing function $Ord^2 \to Ord$ and let $E = \Gamma[R]$. For limit γ, the canonical pairing satisfy $\Gamma[\gamma^2] = \gamma$.

We argue that $L[E] = V$. We verify that for $X \subseteq Ord$, $X \in V$ iff $X \in L[E]$, and by Problem 5.12 the conclusion follows. For a set of ordinals $X \in V$, let $X \in \mathcal{P}(\aleph_\alpha)$, then there is $\gamma \in \aleph_{\alpha+1}$ s.t. $\varphi(\gamma) = X$. i.e. $\{\beta \mid R(\gamma, \beta)\} = X$. Now take δ s.t. $E \cap \delta \supseteq \Gamma[R|_{\aleph_{\alpha+1}}]$, since Γ is definable in $L[E]$, and $E \cap \delta \in L[E]$, $R|_{\aleph_{\alpha+1}} \in L[E]$ and hence $X = \{\beta \mid R|_{\aleph_{\alpha+1}}(\alpha, \beta)\} \in L[E]$. This shows that $L[E] = V$.

Next we show that $L[E]$ is acceptable. Let us pick $X \subseteq \mathcal{P}(\delta)$, say $|\delta| = \aleph_\alpha$. By assumption there is $\gamma < \aleph_{\alpha+1}$ s.t. $\varphi(\gamma) = X$, i.e. $\{\beta < \aleph_{\alpha+1} \mid R(\gamma, \beta)\} = X$. We notice that $\{\gamma\} \times X \subseteq \aleph_{\alpha+1}^2$, moreover, since $|\{\gamma\} \times X| < \aleph_{\alpha+1}$ and $\aleph_{\alpha+1}$ is regular, there is some $\beta < \aleph_{\alpha+1}$ s.t. $\beta \geq \sup \Gamma[\{\gamma\} \times X]$. Thus X can be recovered from in $E \cap \beta$, i.e. contained in $J_{\beta+1}[E]$. Hence, if $X \in J_{\gamma+\omega}[E] \setminus J_\gamma[E]$ for some γ, $\gamma \leq \beta$ and thus there is surjection from δ to γ. $\square$

2 Problem 11.2

Failure of acceptability: Say U is a $< \kappa$ complete ultrafilter on κ, we observe that $L_\kappa[U] = L_\kappa$, as for all $\delta < \kappa$, $x \in L_\delta$, $x \cap U = \emptyset$.

On the other hand, consider the set of countable Silver indiscernibles $I_{\omega_1}^{L[U]} \subseteq \omega_1^{L[U]}$, which exists in $L[U]$ by Lemma 10.31 and Corollary 10.44. Since it can not be in L_κ, it exists in $L_{\gamma+\omega}[U] \setminus L_\gamma[U]$ for some $\gamma \geq \kappa$. But of course there is no surjection $\omega_1^{L[U]} \to \gamma$ in $L[U]$.

Remark: Equivalently, we can work with a subset of ω that codes the minimal 0 mouse. But we cannot work with the set of terms of the Silver indiscernibles over $L_{\aleph_\omega^{L[U]}}$, see [this link](https://math.stackexchange.com/questions/1888063/why-is-0-sharp-not-definable-in-zfc) for clarification.

Weak acceptablility:

By the proof of problem 10.5, we know that $L[U] \models o(<_{L[U]} |_{\mathcal{P}(\rho)}) = |\rho|^+$. Assuming $(\mathcal{P}(\rho) \cap J_{\alpha+\omega}[U]) J_\alpha[U] \neq \emptyset$, $J_{\alpha+\omega}[U] \models o(<_{L[U]} |_{\mathcal{P}(\rho)^{J_\alpha[U]}}) < |\rho|^+$, and this entails the conclusion.

3 Problem 11.3

The statement of the Lemma is wrong, according to errata, the additional hypothesis $\forall x \in U' \exists y \in U', x \in y$ is needed.

(a) For a Σ_1 formula $\exists x \varphi(x, \vec{y})$, the U to U' direction is easy. If $U' \models \exists x \varphi(x, \pi(\vec{b}))$ for some $\vec{b} \in U'^{|\vec{b}|}$, say $U' \models \varphi(a, \pi(\vec{b}))$, take $a_0 \in U'$ s.t. $a \in a_0$ by cofinalness there is $a' \in U$ s.t. $a \in a_0 \subseteq \pi(a')$. Hence $U' \models \exists x \in \pi(a') \varphi(x, \pi(\vec{b}))$, by absoluteness of Δ_0 formulas, $U \models x \in a' \varphi(x, \vec{b})$ and the conclusion follows.

(b) Say $\varphi(\vec{x}) = \forall y_1 \exists y_2 \psi(y_1, y_2, \vec{x})$. For $\vec{b} \in U'^{|\vec{b}|}$, if $U' \models \forall y_1 \exists y_2 \psi(y_1, y_2, \pi(\vec{b}))$, then for all $a \in U$, $U' \models \exists y_2 \psi(\pi(a), y_2, \pi(\vec{b}))$ and by Σ_1 elementariness, $U \models \exists y_2 \psi(a, y_2, \vec{b})$ and hence the conclusion holds.

(c) Say $\varphi(\vec{x}) = \forall v_1 \exists v_2 \supseteq v_1 \psi(v_2, \vec{x})$, assume $U \models \forall v_1 \exists v_2 \supseteq v_1 \psi(v_2, \vec{b})$. For all $a \in U'$, let $a \subseteq \pi(a')$ for $a' \in U$ and thus there is $a'' \in U$ s.t. $U \models \psi(a'', \vec{b})$. Hence $U \models \psi(\pi(a''), \pi(\vec{b}))$ by upward closedness of Σ_1 formula, clearly $a \subseteq \pi(a'')$. This concludes the proof.

4 Problem 11.4

Claim 1 (Claim 11.4.1)**.** *Strengthening Claim 11.17. Under the assumption of the problem, if $\varphi(x_1)$ is a Σ_{n+1}-formula, for $\bar{x}_i = h_{\bar{M}}(n_i, (\vec{\bar{z}}_i, \bar{p}))$ for $0 < i \leq l$ and $n_i < \omega$ and $\bar{z}_i \in [\rho_1(M)]^{<\omega}$. Then let $x_i = h_M(n_i, (\vec{z}_i, p))$ where $\vec{z}_i = \pi(\vec{\bar{z}}_i)$, then*

$$\bar{M} \models \varphi(\bar{x}_1, \ldots, \bar{x}_l) \iff M \models \varphi(x_1, \ldots, x_l)$$

Proof. I can't find the reference of function e on page 230. But the existence of such e is standard S^m_n argument in recursion theory. Hence we have the existence of function e s.t. for Σ_1 formula φ_n,

$$\bar{M} \models \varphi_n(h_{\bar{M}}(n_1, (\vec{\bar{z}}_1, \bar{p})), \ldots, h_{\bar{M}}(n_l, (\vec{\bar{z}}_{l+n}, \bar{p}))) \iff \bar{M} \models \varphi_{e(n)}(\vec{\bar{z}}_1 \ldots \vec{\bar{z}}_{l+n}, \bar{p})$$

And similarly for M.

For simplicity of notation, we assume that n is odd, we have that

$$\begin{aligned}\bar{M} \models \varphi(\bar{x_1},\dots,\bar{x_l}) &\iff \bar{M} \models \exists \bar{z}_{l+1}\dots\forall \bar{z}_{l+n} \in [\rho_1(\bar{M})]^{<\omega}, \varphi_n(h_{\bar{M}}(n_1,(\vec{\bar{z_1}},\bar{p})),\dots,h_{\bar{M}}(n_l,(\vec{\bar{z}}_{l+n},\bar{p})))\\ &\iff \exists \bar{z}_{l+1}\dots\forall \bar{z}_{l+n} \in [\rho_1(\bar{M})]^{<\omega}, \bar{M} \models \varphi_{e(n)}(\vec{\bar{z_1}},\dots,\vec{\bar{z}}_{l+n},\bar{p}) \quad (*)\\ &\iff \bar{M}^{\bar{p}} \models \exists \bar{z}_{l+1}\dots\forall \bar{z}_{l+n} A^{\bar{p}}_{\bar{M}}(e(n),(\vec{\bar{z_1}},\dots,\vec{\bar{z}}_{l+n}))\end{aligned}$$

Notice here in $(*)$ we use the fact that $\bar{p} \in R_{\bar{M}}$. And hence by assumption the same holds for M.

Hence by Σ_n elementarity of $\pi : \bar{M}^{\bar{p}} \to M^p$ the claim is proved. □

With the claim, we proceed as the proof of Lemma 11.16 and the conslusion follows. □

5 Problem 11.7

First we show that $cf^{V[G]}(\omega_2^V) = \omega$. Similar to the definition in Problem 10.24, we say s is a stem of T if s is the longest node in T s.t. for all $t \in T$, $t \supseteq s$ or $t \subseteq s$.

Since the set

$$D_n = \{T \in \mathbb{N} \mid T \text{ has stem } s \text{ longer than } n\}, n \in \omega$$

$$E_\alpha = \{T \in \mathbb{N} \mid \text{the stem of } T, s \text{contains} \beta \geq \alpha\}, \alpha \in \omega_2$$

are dense, hence the $\bigcup\{s \mid s \text{ is the stem of } T, T \in G\}$ is a cofinal in ω_2.

Next we show that Namba forcing preserves ω_1. We every function $\tau : \omega \to \omega_1$ in $V[G]$ is bounded. If $T \Vdash \tau : \omega \to \omega_1$, then we recursively choose $(t_s, T_s, \alpha_s \mid s \in \omega_2^{<\omega})$ s.t. 1. $T_\emptyset = \emptyset$, $t_\emptyset$ 2. $t_s \in T_s$, s is part of the stem of T_s, $T_s \Vdash ran(\tau|_{lh(s)}) \subseteq \alpha_n$ for $\alpha_n < \omega_1$. 3. $\{t_{s \frown \xi} \mid \xi < \omega_2\} \subseteq T_s$ is a family of extensions of t_s of the same length of size $\aleph_2$. For $\alpha < \omega_1$, define

$$T^\alpha = \bigcap\{\bigcup\{T_s \mid h(s) = n, \alpha_s < \alpha\} \mid n < \omega\}$$

For a tree T, define

$$T' = \{t \in T \mid \text{there are } \omega_2 \text{ many nodes below } t\}$$

Similar to the Cantor Bendixon argument, define $T_{\alpha+1} = T'_\alpha$ and $T_\lambda = \bigcap_{\alpha<\lambda} T_\alpha$, $T^* = \bigcap_{\alpha<\omega_3} T_\alpha$. Let $||t||_T = \alpha$ if $t \in T_{\alpha+1} - T_\alpha$, if $t \in T^*$, set $||t||_T = \infty$.

We argue that there is some $\alpha < \omega_1$ s.t. $||\emptyset||_{T^\alpha} = \infty$, otherwise consider the string $c_0 : \omega \mapsto 0$, pick $\alpha > \alpha|_{c_0|_n}$ for all n. Notice that $T_{c_0|_n} \subseteq T^\alpha$ for all n, and thus by construction of T_s, $||t_{c_0|_{n+1}}||_{T^\alpha} < ||t_{c_0|_n}||_{T^\alpha}$, a contradiction to the well-foundedness of ordinals.

Given there is some $\alpha < \omega_1$ s.t. $||\emptyset||_{T^\alpha} = \infty$, consider $(T^\alpha)^*$, it is a perfect tree, $(T^\alpha)^* \leq T$ and $(T^\alpha)^* \Vdash ran(\tau) \subseteq \alpha$. This shows that $V[G] \models \tau_G$ is bounded in ω_1.

Finally, we show that the constraint in Jensen's covering lemma cannot be left out. First observe that $|\omega_2^V|^{V[G]} = \omega_1^{V[G]} = \omega_1^V$ as $\omega_2^V < \aleph_\omega \leq \omega_2^{V[G]}$, which is the first uncountable cardinal in $V[G]$ with cofinality ω. This shows that ω_V is not a cardinal in $V[G]$ and thus $|\omega_2^V|^{V[G]} = \omega_1^{V[G]} = \omega_1^V$.

Apply Namba forcing to L. Since Namba forcing preserves ω_1, a set of ordinals is countable in L iff it is countable in $L[G]$. Consider the set $\aleph_2^L$ in $L[G]$, take f as its cofinal sequence of type omega, since L thinks that there is no countable sequence cofinal in $\aleph_2^L$, any cover $Y \in L$, $Y \supseteq \{f(n) \mid n \in \omega\}$ would be uncontable in L. Thus it would be uncountable in $L[G]$.

For statement (3), consider in $L[G]$ the club set $C = \{f \in [\omega_2^L]^\omega \mid f \text{ is unbounded in } \omega_2^L\}$. This set does not intersect $[\omega_2^L]^\omega \cap L$ $\Box$

6 Problem 11.8

This is essentially Claim 10.16.1 in [[Ch. 10 of Schindler's text book]].

Alternatively, suppose there is forcing $\mathbb{P}$ that adds $0^\sharp$, then $Con(ZFC) \rightarrow Con(ZFC{+}0^\sharp \text{ exists })$, which implies $Con(ZFC) \rightarrow Con(ZFC{+} \text{ inaccessible cardinals exists })$, this is absurd as ZFC and ZFC+inaccessible cardinal is not equiconsistent. $\Box$

7 Problem 11.11

Following the hint of problem 19.12 in Jech, Let $\kappa = \omega_1$. Since $(\kappa^+)^L \leq \omega_2$, $cf((\kappa^+)^L) \leq (\omega_2)$. But as both ω_1 and ω_2 are singular, $cf(\omega_2) = cf((\kappa^+)^L) = \omega$. We pick X, Y two cofinal set of size ω for $\kappa = \omega_1$ and $(\kappa^+)^L$ respectively.

Consider the model $L[X, Y]$, which is a model of ZFC and ω_1 is a singular cardinal in $L[X, Y]$ while $(\kappa^+)^L$ is not a cardinal since models of ZFC thinks successor cardinals are regular. By corollary 11.60 $0^\sharp$ exists in $L[X, Y]$, thus in V.