Solutions to problems in Ch.12

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1 Problem 12.1

Let $(x_i, i < \theta)$ be a sequence of different elements of $\mathcal{P}(\omega)$, we proceed as hinted and take U to be a measure on ω_1 by theorem 12.18. By Problem 10.3 a, ω_1 is measurable in $L[U, (x_i, i < \theta)]$, thus $\theta \leq |\mathcal{P}(\omega)|^{L[x]} < \omega_1$ since $L[U]$ is a model of ZFC. □

2 Problem 12.2

Failure of AD trivially implies the conclusion. So we assume AD.

Consider the following game: player I plays an ordinal $\alpha \in \omega_1$ and player II plays ordinal $n_m \in \omega$ for $m \in \omega$ many times. II wins iff the consequence of the game x satisfies $||x|| = \alpha$.

Then I does not have a winning strategy, but a winning strategy for II would be an injection $\omega_1 \to \omega^\omega$, contradicting the conclusion of problem 12.1. □

3 Problem 12.3

For each formula without parameters $\varphi(x, y, z)$, the set

$$\{a \in \omega^\omega \mid \exists \alpha (x \in a \iff \varphi(x, \alpha, A))\}$$

must be countable as otherwise by replacement this induces an injection $\omega_1 \to \omega^\omega$, contradicting 12.1. Thus since there are only countably many formulas, by AC_ω on reals, $\mathrm{OD}_{\{A\}} \cap \omega^\omega$ is countable.

Let $f : \omega^\omega \to \omega^\omega$ be in $\mathrm{OD}_{\omega^\omega \cup \{A\}}$, then there is formula φ s.t. there is $\alpha_i \in Ord$, $z_j \in \omega^\omega$, for all $x, y \in \omega^\omega$

$$f(x) = y \iff \varphi(x, y, z_1 \dots z_n, A, \alpha_1 \dots \alpha_m)$$

Let z be $\oplus_{1 \leq i \leq n} z_i$, then

$$a \in f(z) \iff \exists y (a \in y \wedge \varphi'(y, z, A, \alpha_1 \dots \alpha_m))$$

for some proper φ' constructed from φ. This shows that $f(z) \in \mathrm{OD}_{\{z,A\}}$.

For the final conclusion, just take $A_x = \omega^\omega \setminus \mathsf{OD}_{\{x,A\}}$. □

4 Problem 12.4

Let f be given by $x \mapsto Th_{\Sigma_1}(x^\sharp)$. The function is ordinal definable as

$$n \in Th_{\Sigma_1}(x^\sharp) \iff L_{\aleph_\omega}[x] \models \varphi_n$$

And $Th_{\Sigma_1}(x^\sharp)$ cannot be in $L[x]$ by indefinability of truth.

Since $\mathbb{C}$ is contained in $L_{\kappa_0}[x]$ where κ_0 is the first Silver indiscernible and $L[x]$ deem κ_0 as inaccessible, all the dense set in $L[x]$ of $\mathbb{C}$ must already be contained in $x^\sharp$ as $\mathcal{P}(\mathbb{C})^{x^\sharp} = \mathcal{P}(\mathbb{C})^{L[x]}$. Thus for each $x \in \omega^\omega$ the dense sets in $L[x]$ must be countable.

Now by AC_ω, there is Φ be a function s.t. $\Phi(n,x)$ is the n-th dense set in $L[x]$ and $\{\Phi(n,x) \mid n \in \omega\}$ enumerates the dense sets in $L[x]$. Fix a canonical enumeration of conditions in $\mathbb{C} = \{s_n \mid n \in \omega\}$. Now we run a generic filter existence argument (Lemma 6.4) in a definable way:

We enumerate $\mathcal{D}_x = \{\Phi(n,x) \mid n \in \omega\}$ and at each step we add the smallest condition in $\mathbb{C}$ that is in $\Phi(n,x)$ to the filter base. Let the consequent generic filter be G_x.

Then $x \mapsto G_x$ is the desired function. □

Remark: It seems the countability of $\mathbb{C}$ is crucial.

5 Problem 12.5

(a) As hinted, consider the game $G_{Wadge}(A,B)$ where I plays x_n and II plays y_n and I wins iff $x \in B \iff y \in A$.

If τ is a winning strategy for I then $x \in B \iff e(\tau * x) \in A$, here $e : \omega^\omega \to \omega^\omega$ is the map taking a real and forming a new real from the even indices. Since the operation $\tau*$ is continuous and taking the even is continuous, this shows that $A \leq_{Wedge} B$. If σ is a winning strategy for II then $o(\sigma * y) \notin A \iff y \in B$ where o is taking the odd indices. □

Identity function witnesses reflexivity and composition of continuous functions entail transitivity.

$\emptyset$ and a set $\emptyset \subsetneq X \subsetneq \omega$ witnesses the failure of symmetry.

(b)We proceed as hinted. Let $A_0 >_{Wadge} A_1 >_{Wadge} \dots$, we show that I has a winning strategy for both $G_{Wadge}(A_{n+1}, A_n)$ and $G_{Wadge}(\omega^\omega - A_{n+1}, A_n)$ for all n. Since otherwise by the argument in (a) and determinacy, $A_{n+1} \leq_{Wadge} A_n$, contrary to assumption. Let σ_n^0, σ_n^1 be the respective winning strategy for the two games.

For $z \in 2^\omega$, we define x_n^z recursively by

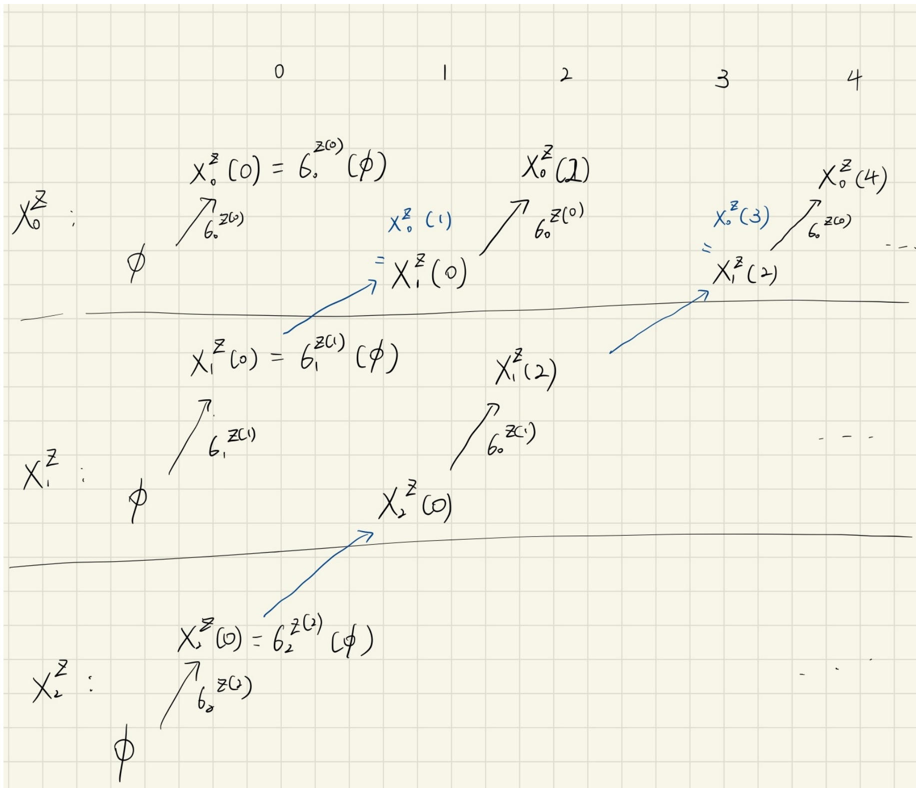
$$x_n^z(2m+2) = \sigma_n^{z(n)}(x_n^z(0), x_{n+1}^z(0), \dots x_{n+1}^z(2m), x_{n+1}^z(2m))$$

and

$$x_n^z(2m+1) = x_{n+1}^z(2m)$$

for all n.

The picture visualizes what's going on:



If follows that $x_n^z = \sigma_n^{z(n)} * x_{n+1}^z$. And it is easy to check that $\{z \in 2^\omega \mid x_0^z \in A_0\}$ is a flip set and by Problem 8.3 is not Lebesgue measurable, thus contradicting Theorem 12.13. $\square$

(c) The constant function is an injection from ω^ω to the continuous functions. Pick a countable dense set D in $\mathbb{R}$, then a continuous function f is determined by $f|_D$, since the set $\{f : D \to \omega^\omega\} \cong \{f : D \times \omega \to \omega\} \cong \{f : \omega \to \omega\}$, there is injection from the continuous functions to ω^ω. By Cantor-Berstein, there is a bijection from $x \in \omega^\omega$ to f_x continuous. Consider the set $J(A) := \{x \mid f_x(x) \notin A\}$, we show that $A <_{Wadge} J(A)$.

First $A \not\geq_{Wadge} J(A)$ as if there is continuous g s.t. for all x, $g(x) \in A \iff x \in J(A)$, say $g = f_x$ then $f_x(x) \in A \iff x \in J(A) \iff f_x(x) \notin A$, a contradiction.

Hence by (a) $A \leq_{Wadge} J(A)$ and we are done. $\square$

(d) $\leq$: Let $x \mapsto g_x$ as the bijection given above. For A s.t. its Wedge rank is α, then $f : x \mapsto ||g_x^{-1}(A)||_{<_{Wadge}}$ is a surjection $\omega^\omega \to \alpha + 1$.

$\geq$: If $f : \omega^\omega \to \alpha$ is a surjection, as hinted, inductively define $A_\nu := J(\{x \oplus y \mid f(x) < \nu, y \in A_{f(x)}\})$ for limit ν and $A_{\alpha+1} := J(A_\alpha)$

It suffice to show that if $\nu < \xi$, then $A_\nu <_{Wadge} A_\xi$, this is by an induction and the successor case is easy. If ξ is limit, $A_\nu \leq_{Wadge} \{x \oplus y \mid f(x) < \xi, y \in A_{f(x)}\} <_{Wadge} A_\xi$.

6 Problem 12.6

(a) Consider the simple game where I plays a real x and II plays a real y, II wins iff $y \in A_x$. Then the winning strategy for II is such a choice function.

(b) We proceed as hinted. Let

$$A \in U \iff \text{I has a winning strategy in } G(\{f \in (\omega^\omega)^\omega \mid ran(f) \in A\})$$

This U of course is upward closed and contains $[\omega^\omega]^{\aleph_0}$. For any A, I has a winning strategy in $G(\{f \mid ran(f) \in A\}) \iff G(\{f \mid ran(f) \in A^c\})$. By $AD_\mathbb{R}$ U thus contains exactly one of an element and its complement. If there is $A_n, n \in \omega$ s.t. $A_n \in U$

To show that U is countably complete, let $A_n, n \in \omega$ be a sequence of sets in U, by (a) we may have τ_n winning strategy for I in game $G(\{f \mid ran(f) \in A_n\})$. We consider the following winning strategy τ for I in $G(\{f \mid ran(f) \in \bigcap_n A_n\})$.

Let $(-,-) : \omega^2 \to \omega$ be a bijection such that $(n,m) < (n,m+1)$. At step $(n,m+1)$ for I, given previous plays n_i, I plays what τ_n yields on the play

$$(n_{(n,0)}, \oplus_{(n,0)<i<(n,1)} n_i, \ldots, n_{(n,m)}, \oplus_{(n,m)<i<(n,m+1)} n_i)$$

i.e. I plays τ_n viewing the previous plays intermediate $(n,i),(n,i+1)$ as a single move by his opponent at move i.

Then since τ_n wins I $G(\{f \mid ran(f) \in A_n\})$, $\tau * x \in \{f \mid ran(f) \in A_n\}$ for all n and thus τ is a winning strategy for I.

$\{a \in [\omega^\omega]^{\aleph_0} \mid x \in a\} \in U$ is obvious: I wins by playing x at his first move.

Finally, let $A_x \in U$ for $x \in \omega^\omega$. Let τ_x be a winning strategy for the game on A_x. The strategy is similar to countable closedness. For each existing play x_n, I makes sure to apply τ_{x_n} infinitely often in the future, regarding intermediate plays between two application of τ_{x_n} as a single move by his opponent. $\square$

7 Problem 12.7

(a) First we observe that for each continuous function f is OD_{ω^ω}, as the bijection developed in Problem (c) is OD_{ω^ω}.

Next, for arbitrary $\alpha = ||A||_{<_{Wadge}} < \Theta_0$, pick $f : \omega^\omega \to \alpha + 1$ in OD_{ω^ω} and we define $A_\nu, \nu \leq \alpha$ as in Problem 12.5. Then the sequence $||A_\alpha|| \geq \alpha$ and thus there is g continuous $A = g^{-1}(A_\alpha)$. Thus

$$x \in A \iff g(x) \in A_\alpha$$

This means that $A \in OD_{\omega^\omega}$ $\square$

(b) This is the exact same argument parametrized by B $\square$.

8 Problem 12.8

Assume $\mathrm{AD}_\mathbb{R}$, we argue that for every $B \subseteq \omega^\omega$, $\mathcal{P}(\omega^\omega) \not\subseteq \mathsf{HOD}_{\{B\}\cup\omega^\omega}$. Since by Problem 12.3 there is non-empty $(A_x, x \in \omega^\omega) \in \mathsf{HOD}_{\{B\}\cup\omega^\omega}$ without choice function. But working in V, by $\mathrm{AD}_{\mathbb{R}}$ and Problem 12.6 there is a choice function $\omega^\omega \to \omega^\omega$ in V, since being a choice function for a given family of sets is absolute, hence $\mathcal{P}(\omega^\omega) \not\subseteq \mathsf{HOD}_{\{B\}\cup\omega^\omega}$. By Problem 12.7 this means that the length of the Solovay sequence is not a successor.

Since assuming AD, the model $\mathsf{HOD}_{\{B\}\cup\omega^\omega}$ is a model of AD there is non-empty $(A_x, x \in \omega^\omega) \in \mathsf{HOD}_{\{B\}\cup\omega^\omega}$ without choice function, by Problem 12.6 it is not a model of $\mathrm{AD}_{\mathbb{R}}$, this concludes the proof.

9 Problem 12.10

(a) Let α be an ordinal countable in L s.t. $J_\alpha \models ZFC^-$. Work in L and since L knows J_α is countable, L can add a Cohen generic real into J_α to form M. Then $Ord(M) = \alpha$ and $(M \cap L \cap \mathcal{P}(\omega)) \setminus J_\alpha \neq \emptyset$.

(b) Assume for contradiction that α is not an L-cardinal, then there is κ an L-cardinal s.t. $\kappa < \alpha < (\kappa^+)^L$. Pick $f : \kappa \to J_\alpha$ in L surjective, define $E \subseteq \kappa^2$ s.t. $\xi_1 E \xi_2$ iff $f(\xi_1) \in f(\xi_2)$ where $E \in L$. By the assumption $\mathcal{P}(\kappa) \cap L \subseteq M$, $E \in M$ and we apply Mostowski Collapse in M to κ, E and obtain that $J_\alpha \in M$. This contradicts the fact that $\alpha = M \cap Ord$. $\square$

10 Problem 12.11

Since $\alpha \geq \kappa + \omega$ is x-admissible, by Problem 5.28 $J_\alpha[x]$ does transitive collapse correctly. Hence the κth iterate of $0^\sharp$ exists in $J_\alpha[x]$, as well as the $\kappa + 1$th iterate of $0^\sharp$, call them $\mathcal{M}_\kappa, \mathcal{M}_{\kappa+1}$ respectively. Since $\kappa \in \mathcal{M}_\kappa$, $\mathcal{P}(\kappa)$ is the same in $\mathcal{M}_L$ as in L. Hence $\mathcal{P}(\kappa)^L \subseteq J_\alpha[x]$.

That α is an L cardinal follows directly from Problem 12.10 (b). $\square$

11 Problem 12.12

We argue as in the proof of theorem 10.11. Consider the similar game G_s but instead played on $[\omega]^{<\omega}$ and ω repectively. Similarly, we show that:

Claim 1 (Claim 12.12.1)**.** *I does not have a winning strategy for G_t in M.*

Proof. We work in M. Say σ is a winning strategy for I. Notice that by how the game is played, a winning strategy of I would not depend on his previous move, and hence σ can be viewed as a function taking input from $[\omega]^{<\omega}$. As $\sigma(t \frown s) \in U$ for all $s \in [\omega]^{<\omega}$, by selectivity of U, by problem 9.3 (b), we thus have $Y \in U$ s.t. for all strictly increasing $s \in \omega^{<\omega}$, if $ran(s) \subseteq Y$ then $s(n) \in \sigma(t \frown s|n)$. We say that Y selects the system $\sigma(t \frown s)$. We pick $(t', Y') \leq (t, Y \setminus max(t))$ where $(t', Y') \in D$. We have $t' - t \subseteq Y$.

We argue that by playing as II the moves $t' - t$ in the first few rounds, he defeats the strategy σ. Let $t' - t = \{n_m \ldots n_l\}$. As σ would respond to the play $t \frown \{n_m \ldots n_{m+i}\}$ with $\sigma(t \frown \{n_m \ldots n_{m+i}\})$, thus $n_{m+i+1} = t'(m+i+1) = (t'-t)(i+1) \in \sigma(t \frown \{n_m \ldots n_{m+i}\})$ and hence II's next move is still legit.

Argue inductively in the above way, II can play $t' - t$ in the first l rounds and since $(t', Y') \in D$, she wins the game. $\square$

Hence, as G_t is a closed game, II has a winning strategy in M. This is a winning strategy in V also as all the plays legit by I in V are already in M. Let τ be such a winning strategy.

Similarly, we have that

Claim 2 (Claim 12.12.2)**.** *If t is realizable, then there is $Y_s^t \in U$ s.t. for all $\lambda \in Y_s^t$ there is X s.t. playing λ, X as the respective next move of II,I, $X \in U$ and II played according to τ.*

The aim of this claim is to enable us to design a sequence of plays by II so that the consequence lands in G, while the plays respects τ and thus the consequence lands in D.

Associate Z_s to s if $(s, Z_s) \in D$ and otherwise $Z_s = \omega$. Let Y_t^* be the the intersection of $Y_s^t, s \subseteq t$. Finally let W_0 select Z_s and W_1 select Y_t^*. Then the set $W_0 \cap W_1$ and thus agrees with x after some m. Say $x = \{x_n, n \in \omega\}$ and $W_0 \cap W_1 \supseteq \{x_m, x_{m+1} \dots\}$. Then in the game G_s where $s = \{x_0 \dots x_m\}$, any initial $\{x_0, \dots, x_n\}$ is realizable. Thus by the same argument as in the book, there is $)\{x_0 \dots x_n\}, Z_{x|_n}) \in D \cap G$. $\square$

12 Problem 12.13

Claim 3 (Claim 12.13.1)**.** *Mathias forcing has pure decision: For any formula $\varphi(\vec{\tau})$ and condition (s, A), there is $B \subseteq A$ s.t. (s, B) decides $\varphi(\vec{\tau})$*

See for instance Jech Lemma 26.34. I don't see a proof in the style of Claim 10.7. Though we have that for selective ultrafilter U the following property holds: for each n, k and $F : [\omega]^n \to k$ there is $X \in U$ homogeneous. However, this does not seem strong enough to allow us to run the proof of Claim 10.7.

We show that if A is Solovay over $M[s]$ then A is Ramsey for $s \in Ord^\omega$. Say

$$x \in A \iff V[s][x] \models \varphi(x)$$

Consider Mathias forcing $\mathbb{M}$ in $M[s]$ and pick $(\emptyset, X)$ deciding $\varphi(\dot{x})$ by Claim 12.13.1, here $\dot{x}$ is the canonical name of the generic real. Say $(\emptyset, X) \Vdash \varphi(\dot{x})$. As κ is still inaccessible in $V[s]$, there is $\mathbb{M}$ generic G over $V[s]$. we argue that $[\dot{x}_G]^\omega \subseteq A$, for arbitrary $y \subseteq \dot{x}_G$, $y \subseteq X$ and the filter corresponding to y contains $(\emptyset, X)$, thus $V[s][y] \models \varphi(y)$, i.e. $y \in A$. $\square$