Solutions to problems in Ch.13

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1 Problem 13.1

The solution goes as the hint in D.A. Martin's Exercise 8.3.2

We fix an enumeration of ω^ω: $(x_\alpha, \alpha < 2^{\aleph_0})$. Let $(M_\alpha, \pi_{\alpha,\beta}, \alpha < \beta \in Ord)$ be the linear iteration of V by U, the normal κ complete ultrafilter on κ. Let $N = M_{2^{\aleph_0}}$. Let κ_α be $\pi_{0,\alpha}(\kappa)$.

For $s \in \omega^{<\omega}$, $\alpha < 2^{\aleph_0}$, define $j^s_{\alpha,\beta} : N \to N$ elementary by:

$$j^s_{\alpha,\alpha+1} = \begin{cases} \pi_{\omega\cdot\alpha,\omega\cdot\alpha+1} & \text{if } s \subseteq x_\alpha, x_\alpha \notin A \\ id & \text{otherwise} \end{cases}$$

The case $j^s_{\gamma,\alpha+1}$ where $\gamma < \alpha$ is defined by $j^s_{\gamma,\alpha} = j^s_{\alpha,\alpha+1} \cdot j^s_{\gamma,\alpha}$. And the limit case is by directed limit.

Define $M_\emptyset = V$, $M_s = N$ for $s \neq \emptyset$. For $s \subseteq t$ s.t. $lh(t) = lh(s) + 1$,

$$k_{s,t} = \begin{cases} \pi_{0,\aleph_0} & \text{if } s = \emptyset \\ j^s_{0,2^{\aleph_0}} & \text{otherwise} \end{cases}$$

The other $k_{s,t}$ are defined by commutivity. We notice the effect of $k_{s,t}$ on the critical points where $lh(t) = lh(s) + 1$:

$$k_{s,t}(\kappa_{\omega\cdot\alpha+k}) = \begin{cases} \kappa_{\omega\cdot\alpha+k+1} & \text{if } s \subseteq x_\alpha, x_\alpha \notin A \\ \kappa_{\omega\cdot\alpha+k} & \text{otherwise} \end{cases}$$

Now we show that $(M_s, k_{s,t}, s \subseteq t \in \omega^{<\omega})$ is a embedding normal form for A.

If $x_\alpha \notin A$, then for each n, $k_{0,x|_n}(\kappa_\alpha) > k_{0,x|_{n+1}}(\kappa_\alpha)$ as $\pi_{\omega\cdot\alpha}$ is acted in $k_{x|_n,x|_{n+1}}$. Hence M_{x_α} is ill founded.

If $x_\alpha \in A$, let $n_\beta = 0$ if $x_\beta \in A$, otherwise $n_\beta = max\{n \mid x_\beta|_n = x_\alpha|_n\}$. M_{x_α} can be decomposed as $dir\lim(\pi^{n_\alpha}_{\omega\cdot\alpha}(N), \alpha < 2^{\aleph_0})$, which is well-founded. □

2 Problem 13.2

If $x \notin A$, then the tree $T_x := \{t \in \alpha^{<\omega} \mid (x|_{len(t)}, t) \in T\}$ is well-founded. Then for each n,

$$||[id]_{\mu_{x|_n}}||_{\pi_{\mu_{x|_n}}(T_x)}$$

is an ordinal. To show that the direct limit is ill-founded, it suffice to show that

$$||[id]_{\mu_{x|_{n+1}}}||_{\pi_{\mu_{x|_{n+1}}}(T_x)} < \pi_{\mu_{x|_n},\mu_{x|_{n+1}}}(||[id]_{\mu_{x|_n}}||_{\pi_{\mu_{x|_n}}(T_x)}) = ||\pi_{\mu_{x|_n},\mu_{x|_{n+1}}}([id]_{\mu_{x|_n}})||_{\pi_{\mu_{x|_{n+1}}}(T_x)}$$

Suppose $||\pi_{\mu_{x|_n},\mu_{x|_{n+1}}}([id]_{\mu_{x|_n}})||_{\pi_{\mu_{x|_{n+1}}}(T_x)} = \beta$, we have $\pi_{\mu_{x|_n},\mu_{x|_{n+1}}}([id]_{\mu_{x|_n}})(t) = [id]_{\mu_{x|_n}}(t|_n) = t|_n$ for arbitrary $t \in T_{x|_{n+1}}$. Thus there is $X \in \mu_{x|_{n+1}}$ s.t. for all $t \in X$, $||t|_n||_{\pi_{\mu_{x|_{n+1}}}} = \beta$, but as for any $t \in X$,

$$||t||_{\pi_{\mu_{x|_{n+1}}}} < ||t|_n||_{\pi_{\mu_{x|_{n+1}}}}$$

we have the desired result. □

3 Problem 13.3

(b) to (a). Fix T on $\omega \times \alpha$ s.t. $A = p[T]$ and μ_s is a $< \delta^+$ closed ultrafilter on T_s. We define the following system of elementary embedding: $(M_s, \pi_{s,t}, s \subseteq t \in \omega^{<\omega})$. Where $M_s = Ult(V, \mu_s)$. Then by problem 13.2 we have $x \in A$ iff $dir \lim_{n<\omega}(M_{x|_n})$ is well-founded. It suffice to check that thus defined, the embedding normal form is $2^{\aleph_0}$ closed and its additivity is bigger than δ.

As $crit(\pi_{\mu_s}) \geq \delta^+$, $\pi_{s,t}|_{\delta+1} = id$ for any $s \subseteq t \in \omega^{<\omega}$. To show that the embedding normal form is $2^{\aleph_0}$ closed, either μ_s is trivial and thus $Ult(V, \mu_s) = V$, thus trivially $2^{\aleph_0}$ closed; or μ_s is not trivial in which case let κ be the critical point of π_{μ_s} and thus $2^{\aleph_0} < \kappa$, $Ult(V, \mu_s)$ is κ closed.

(a) to (b). Let $(M_s, \pi_{s,t}, s \subseteq t \in \omega^{<\omega})$ be the assumed embedding normal form. We work as hinted. Let $\alpha_x^n, n < \omega$ be the ordinals witnessing M_x is ill-founded in $x \notin A$; let T be the Windßus tree for the embedding normal form and $\alpha_s(x)$ be as defined in the proof theorem 13.2. Define the following ultrafilter μ_s on T_s: for $X \subseteq T_s$,

$$X \in \mu_s \iff (\pi_{\emptyset,s}(\alpha_\emptyset) \ldots \alpha_s) \in \pi_s(X) \subseteq \pi_s(T_s) = \{f \mid (s, f) \in \pi_s(T)\}$$

Let $\sigma_s : V \to Ult(V, \mu_s)$ be the generated elementary embedding. The conclusion follows from the following claims:

(1) If $s \subseteq t \in \omega^{<\omega}$, then μ_s, μ_t is cohere. This is because

$$\begin{aligned}(\pi_{\emptyset,s}(\alpha_\emptyset), \ldots \alpha_s) \in \pi_s(X) &\iff (\pi_{\emptyset,t}(\alpha_\emptyset), \ldots \pi_{s,t}(\alpha_s)) \in \pi_t(X) \\ &\iff (\pi_{\emptyset,t}(\alpha_\emptyset), \ldots \pi_{s,t}(\alpha_s), \ldots \alpha_t) \in \pi_t(\{f \in T_t \mid f|_{|s|} \in X\})\end{aligned}$$

(2) There is the factor elementary embedding $k_s : Ult(V, \mu_s) \to M_s$ s.t. $k_s \cdot \sigma_s = \pi_s$.

Just let $k_s : \sigma_s(F)((\pi_{\emptyset,t}(\alpha_\emptyset), \ldots \pi_{s,t}(\alpha_s), \ldots \alpha_t)) \mapsto \pi_s(F)((\pi_{\emptyset,t}(\alpha_\emptyset), \ldots \pi_{s,t}(\alpha_s), \ldots \alpha_t))$ for $F : T_s \mapsto V$.

(3) If $x \in A$, then

$$dir \lim_{n<\omega}(ult(V, \mu_{x|_n}), \pi_{\mu_{x|_n},\mu_{x|_m}}, n < m < \omega)$$

is well founded.

This is because by point (2), we have for $s \subseteq t \in \omega^{<\omega}$, $\pi_{s,t} \cdot k_s = k_t \cdot \sigma_{s,t}$. This gives for each $Ult(V, \mu_s)$, a map $\pi_{s,x} \cdot k_s : Ult(V, \mu_s) \to M_x$ satisfying $\pi_{s,x} \cdot k_s = \pi_{t,x} \cdot k_t \cdot \sigma_{s,t}$. Hence this gives an elementary embedding $j : dir \lim_{n<\omega}(ult(V, \mu_{x|_n}), \pi_{\mu_{x|_n}, \mu_{x|_m}}, n < m < \omega) \to M_x$, witnessing the well-foundedness of the directed limit.

(4) Each μ_s is $< \delta^+$ closed, otherwise say ξ is the least cardinal s.t. μ_s is $< \xi$ closed. Then there is a partition $X_i, i < \xi$ of T_s s.t. $X_i \notin \mu_s$. Define $a : x = f \mapsto i$ if $f \in X_i$ for $f \in T_s$. We thus have $\xi \leq [a]_{\mu_s} < \sigma_s(\xi)$. Finally, as $\pi_s(\xi) \geq \sigma_s(\xi)$ and $\xi \leq \delta$, this contradicts the fact that $\pi_s|_{\delta+1} = id$.□

4 Problem 13.4

(1) Assume that $A = \{x \mid \exists y, x \oplus y \in B\}$, assume that T on $\omega \times \alpha$ and the system $\mu_s, s \in \omega^{<\omega}$ witnesses the δ-homogeneously Susliness of B.

Fix a bijection $\Gamma : \omega \times \alpha \to \alpha$. We first define the tree T^* on $\omega \times \alpha$ as $t^* \in T^*$ iff the map t defined by $t(2n, \xi) = t^*(n, \Gamma(n, \xi))$ and $t(2n, \xi) = t^*(n, \xi)$ is in T. Then $p[T] = A$.

Next for any $y \in \omega^\omega$, $x \notin A$, let $\alpha_n^{x,y}$ witness the ill-foundedness of $dir \lim_{n<\omega}(Ult(V, \mu_{s_1 \oplus s_2}), s_1 \subseteq x, s_2 \subseteq y)$. Let γ be some ordinal greater than all of them. Fix an enumeration of $\omega^{<\omega}$ s.t. $s_i \subseteq s_j$ then $i < j$. Define the following tree on $\omega \times \gamma^{<\omega}$: $(s, (\alpha_0 \ldots \alpha_{k-1})) \in S$ iff for all $i < j < k$, if $s_i \subsetneq s_j$ then

$$\delta^+ < \alpha_j < \pi_{\mu_{s|lh(s_i) \oplus s_i, s|lh(s_j) \oplus s_j}}(\alpha_i)$$

Then we have $x \in p[S]$ iff $x \notin A$.

Next we show that for any $\mathbb{P} \in H_{\delta^+}$, $\Vdash^{\mathbb{P}} p[T^*] \cup p[S] = \omega^\omega$. By the argument in problem 10.19, we have for $\mathbb{P}$ name τ, $\pi_{\mu_s}^{V[G]}(\tau_G) = (\pi_{\mu_s}(\tau))_G$, specifically for element a in V, $\pi_{\mu_s}^{V[G]}(a) = \pi_{\mu_s}(a)$.

Let G be a generic filter, work in $V[G]$. If $x \notin p[T^*]$, then for all $y \in \omega^\omega$, $x \oplus y \notin B$, i.e. for any $y \in \omega^\omega$, there is $\alpha_n^{x,y}$ witnessing the ill-foundedness of $dir \lim_{n<\omega}(Ult(V, \mu_{s_1 \oplus s_2}), s_1 \subseteq x, s_2 \subseteq y)$. Thus $(x|_n, (\alpha_0^{x,y}, \ldots, \alpha_{n-1}^{x,y})), n \in \omega$ would be an infinite branch in S.

(2) Let A be arbitrary $\mathbf{\Sigma}_2^1$ set, and write $A = \{x \mid \exists y \in \omega^\omega, x \oplus y \in B\}$ where B is $\mathbf{\Pi}_1^1$.

From the proof of theorem 13.3, we obtain that B has a $2^{\aleph_0}$ closed embedding normal form which has additivity greater than δ for arbitrary $\delta < \kappa$. By the proof of problem 13.3 there is tree T and system $\mu_s, s \in \omega^{<\omega}$ s.t. each μ_s is $< \kappa$ closed. By the proof of problem 13.4, A is thus $< \kappa$ universally Baire.

(3) Prove by induction on n that if there are n many measurable cardinals $\delta_1 < \cdots < \delta_n$ under κ, then all $\mathbf{\Pi}_{n+1}^1$ sets have a $2^{\aleph_0}$ closed embedding normal form whose additivity is bigger than δ_1. The The induction step is by theorem 13.6. For the base case: it is clear that the construction in theorem 13.3 is has additivity bigger than δ_n^+.

Then apply problem 13.3 and problem 13.4 and we are done.

5 Problem 13.5

Let $A = p[T]$, $A^c = p[U]$ where $\Vdash_{\mathbb{P}} p[U] \cup p[T] = \omega^\omega$ for all $\mathbb{P} \in H_{\delta_1}$. Then for g generic for $\mathbb{P} \in H_{\delta_1}$, $V[g] \models A^* \neq \emptyset$ iff $V[g] \models T$ is ill-founded iff $V \models T$ is ill-founded iff $V \models A \neq \emptyset$.$\square$