Solutions to problems in Ch.9

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1 Problem 9.1

(a) May assume that $X_0 \supseteq X_1 \supseteq \dots$ by taking intersections.

Let

$$f(n) = \begin{cases} 0 & \text{if } \forall m, n \in X_m \\ 1 & \text{if } n \notin X_0 \\ m+2 & \text{where } m = max\{m \mid n \in X_m\} \end{cases}$$

Since F is a p-point, there is $Y \in F$ s.t. F is constant or finite to one on Y. Notice f cannot be constant on $n \geq 1$, since then Y would be disjoint from some element not in F. If f is constant on 0, then $Y - X_n = \emptyset$ for all n. If f is finite to one on Y, then $Y - X_n = \bigcup_{0<m\leq n+2} X_m$ which is finite. $\square$

(b) Let $Y_n = \bigcup_{m\geq n} X_m$ and apply (a). Then the case that f is constant on 0 cannot happen as $\bigcap_n Y_n = \emptyset$. Then there is Y s.t. f is finite to one on Y, then $Y \cap X_n$ is finite for all n.

If in addition F is selective, let

$$\begin{aligned} f : Y &\to \omega \\ m &\mapsto n \text{if } m \in Y \cap X_n \end{aligned}$$

By F is selective, there is $X \subseteq Y$ s.t. f is one-to-one on X, then the conclusion holds for X. $\square$

2 Problem 9.2

(a) Consider the partition $\{[n, n+1) \mid n \in \mathbb{Z}\}$ of $\mathbb{R}$. If none of $f^{-1}[n, n+1)$ is in F, apply 9.1 b and we are done.

Otherwise if say $f^{-1}[n, n+1)$ is in F, divide $[n, n+1)$ into infinitely many intervals and repeat the process.

Eventually, either we get an interval I with partition I_n s.t. $f^{-1}(I) \in F$ and $f^{-1}(I_n) \notin F$ for all n and we are done by 9.1 (b), or we have $I_0 \supseteq I_1 \dots$ s.t. $f^{-1}(I_n) \in F$. May assume that $\bigcap I_n = \emptyset$.

Apply 9.1 (a) there is $Y \in F$ s.t. $Y - f^{-1}(I_n)$ is finite for all n, then in each $I_n - I_{n+1}$ there are only finitely many elements in $f[Y]$ and we are done. $\square$

(b)For any $f : \omega \to \omega$ monotone, consider $g(n) = m$ where $f(m) \leq n < f(m+1)$, this is a finite to one function and since F is a q-point, there is $X \in F$ s.t. $g|_X$ is one-to-one this means that $|X \cap f(n)| \leq n$. $\square$

3 Problem 9.3

Remark: in the terminology of [Mathias' paper](https://www.sciencedirect.com/science/article/pii/0003484377 this shows that a Ramsey ultrafilter is a happy family.

(a) Suffice to show that if $X_0 \supseteq X_1 \supseteq \dots$ for $X_i \in F$, then there is $Y \in F$ s.t. $Y = \{a_0 < a_1 < \dots\}$ and $a_{n+1} \in X_{a_n}$.

By 9.1(a) let $Z \in F$ be s.t. $Z - X_n$ is finite for all n, define

$$g : \omega \to \omega$$
$$n \mapsto max\{Z - X_n\}$$

Case 1: $Y_n = (g(n), g(n+1)] = \emptyset$ for cofinitely many n, this means that $g(n) = g(n+1)$ for $n \geq m$ for some m. i.e. $Z \cap X_m$ is constant for m large enough. Then let Y be $Z \cap X_m$ and we are done.

Case 2: For infinitely many n, $Y_n = (g(n), g(n+1)] \neq \emptyset$. Then we may assume that $Y_n = (g(n), g(n+1)]$ is not empty for all n by redefining g. By 9.1 (b) there is Z' s.t. there is exactly one $m \in Z'$ s.t. $g(n) \leq m < g(n+1)$. Say $Z' = \{m_0 < m_1 < \dots\}$, then one of $\{m_{2n} \mid n \in \omega\}, \{m_{2n+1} \mid n \in \omega\}$ is in F, call it Z^*. Let $Y = Z \cap Z^*$, then as $m_{n+2} \in X_{g(n)+1} \subseteq X_{m_n}$, we are done. $\square$

(b) Let $Y_n = \bigcap\{X_s \mid max(s) \leq n\}$, by (a) fine f ::;$\omega \to \omega$ s.t. $f(n+1) \in Y_{f(n)}$. Then let $Z = \{f(n) \mid n \in \omega\} \in F$. s is s.t. $ran(s) \subseteq Z$, then $s(n) = f(m+1)$ for some $m+1 \geq n$. i.e. $max(s|_n) \leq f(m)$ and hence $s(n) = f(m+1) \in Y_{f(m)} \subseteq X_{s|n}$. $\square$

4 Problem 9.4

Proceed as hinted, let $\{X_n^\alpha \mid n < \omega\}$ $\alpha < \omega_1$ be an enumeration of all infinite partition of ω.

Recursively construct $Y_\alpha, \alpha < \omega_1$ s.t. $Y_\beta - Y_\alpha$ finite if $\beta > \alpha$, and

$$Y_{\alpha+1} = \begin{cases} Y_\alpha \cap X_n^\alpha \text{ for some } n \text{s.t.} Y_\alpha \cap X_n^\alpha \text{ is infinite.} & \text{if such } n \text{ exists} \\ \text{some set s.t. } |Y_{\alpha+1} \cap X_n^\alpha| \leq 1 & \text{otherwise} \end{cases}$$

Let U be the filter generated by $Y_\alpha, \alpha < \omega_1$. U is an ultrafilter since for arbitrary infinite and coinfinite set $X \subseteq U$, it is enumerated as some X_i^α, then Y_α either is contained in X or intersectes with X finitely.

The for any partition of ω into X_i^α where $X_i^\alpha \notin U$, then for Y_α, clause 1 cannot happen and thus $Y_{\alpha+1}$ satisfies the requirement.$\square$

5 Problem 9.6

(a) Consider the following family of dense set for $f \in \omega^\omega$, $n \in \omega$:

$$D_{f,n} = \{t \in \mathbb{P} \mid \exists m > n, t(m) > f(m) \wedge m \in dom(t)\}$$

(b) For any nice name $\tau \in (Fn(\alpha, \omega))^{V[G]}$, by c.c.c. τ is a name of $Fn(\omega \times M, \omega)$, $M \subseteq \alpha$ is countable. Since $Fn(\omega \times M, \omega)$ and $\omega^{<\omega}$ are forcing equivalent, may assume τ is $\omega^{<\omega}$ name. Let $\{p_i \mid i \in \omega\}$ enumerate $\omega^{<\omega}$, find $q_i \leq p_i$ s.t. $q_i \Vdash \tau(i) = j_i$. Define $g : i \mapsto j_i$, g is a function in V. Since there is no p s.t. $p \Vdash \forall m > n, \tau(m) > g(m)$ as there is an extension of $p = p_i$ that forces $p\tau(i) = j_i$ by construction. Hence τ_G is not dominating real. □

6 Problem 9.7

$V[G] \models \aleph_1 = b < \alpha$: Since $Fn(\alpha, 2)$ does not add dominating real, $V \cap \omega^\omega$ is an unbounded family in $V[G]$ of size ω_1.

$V[G] \models \alpha \leq d$: Let $F \subseteq (\omega^\omega)^{V[G]}$ be a family of size $|F| < \alpha$, we show that it is not a dominating family. Let $h : |F| \times \omega \to \omega$ be such that $\{\lambda n.h(\beta, n) \mid \beta < \alpha\} = F$. Let τ be a name of h, then there is $W_0 \subseteq \alpha$ s.t. $|W_0| \leq |F|$ and τ is a $Fn(W_0, w)$ name. Fix $W_0 \subseteq W \subseteq \alpha$ s.t. $|\alpha - W| = \aleph_0$. We hence have

$$Fn(W, 2) \times Fn(\omega, 2) \cong Fn(\alpha, 2)$$

And $V[G] = V[H][K]$ where H, K are generic filter of $Fn(W, 2)Fn(\omega, 2)$ respectively. As $h \in V[H]$, there is an unbounded real over F in $V[H][K]$. Hence F is not dominating. □

7 Problem 9.8

$$D_f := \{(x, n) \mid \exists n(x, n) \leq (f', n) \wedge \forall m \geq nf'(m) = f(m)\}$$

is dense.

8 Problem 9.9

(a) It follows from the fact that any two condition $(s, X), (s, Y)$ is compatible. Hence any antichain in $\mathbb{M}$ gives rise to an antichain in $2^{<\omega}$, which we know is c.c.c.

(b)

$$D_X := \{(s, Y) \mid Y \subseteq X\}$$

is dense for all $X \in F$.

9 Problem 9.10

(a) Let X be a set s.t. neither X or its complement is in $\mathcal{F}$. Let x be the Mathias generic real and define c as $c(n) = 1$ iff the n-th element of x is in X.

We show that c is Cohen generic, that is to show for each D dense in Cohen forcing,

$$\{(s, Y) \mid (s, Y) \Vdash \exists n, \dot{c}|_n \in D\}$$

is dense in Mathias forcing. i.e. for any D dense in Cohen forcing, (s, Y) we show that there is $t \in D$ and $(s', Y') \leq (s, Y)$ forcing $t \subseteq \dot{c}$.

Now given D and (s, Y), define t_0 as $t_0(n) = 1$ iff the n-th element of s is in X. Take $t \leq t_0$ in D and since Y intersects both X and $\omega - X$ infinitely, we can design $(s', Y') \leq (s, Y)$ s.t. $t(n) = 1$ iff the n-th element of s' is in X. This would force $t \subseteq \dot{c}$.

(b) We can show that the 9.1 (b) is actually an equivalence, both for p-point and selective filter.

If it is not a p-point, pick a partition $X_n, n \in \omega$ of ω s.t. $X_n \notin F$ and any $X \in F$ does not intersect every X_n finitely. If x is the Mathias generic real, then $x - X_n$ is finite by a genericity argument, let $X^*_m, m \in \omega$ enumerate those X_n s.t. $X_n \cap x \neq \emptyset$. Define $g : m \mapsto |X^*_m \cap x| - 1$. We show that this is a Cohen real.

For any (s, X), let $X_{n_1} \dots X_{n_m}$ be the sets that intersects s, and hence $\forall n' \geq n_m + 1, X'_n \cap s = \emptyset$. Define $t : m \to \omega$ as $i \mapsto |X_{n_i} \cap s| - 1$. Find $t' \supseteq t$ in D. Notice X intersects infinitely many X_n infinitely, let $Y_m, m \in \omega$ enumerate those X_n where n is greater than n_m. Hence we can find $s' \supset s$ s.t. for $i \in dom(t') \setminus dom(t)$, $t' : i \mapsto Y_{i+1-|dom(t)|}$. Now let X' be X minus all X_n that intersects s'. Then (s', X') forces $t' \subseteq \dot{c}$.

If it is not a q-point, the argument is essentially the same except that in the generic argument, you look at X_n whose intersection with X is sufficiently large instead of infinite. $\square$

10 Problem 9.11

Pick a partition $X_n, n \in \omega$ of ω s.t. $X_n \notin F$ and any $X \in F$ does not intersect every X_n finitely. If x is the Mathias generic real, then $x - X_n$ is finite, let $X^*_m, m \in \omega$ enumerate those X_n s.t. $X_n \cap x \neq \emptyset$. Define $g : m \mapsto max(X^*_m \cap x)$. We show that this is a dominating real.

For any (s, X) and $f \in \omega^\omega \cap V$, let m be s.t. $\forall n \geq m, X_n \cap s = \emptyset$. We have that

$$Y = \bigcup_{n \geq m} X_n - \{\text{the first } f(n)\text{-th element of} X_n\} \in U$$

Since $\bigcup_{n \in \omega} \{\text{the first } f(n)\text{-th element of} X_n\} \notin U$. Then (s, Y) thinks $\dot{g}$ dominates f. $\square$